# Quantitative Macroeconomics

## HH Heterogeneity: Transition Dynamics and Aggregate Fluctuations

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# Introduction

- At this point, we have only focus on the **Stationary Equilibrium**.
- Many questions involve solving the model beyond the Steady-State Stationary Equilibrium.
- **Aggregate Uncertainty:**
  - How the Aiyagari economy reacts to aggregate shocks.
  - Does heterogeneity matters to the business cycles?
- **Transitions Dynamics:**
  - How long it takes to the economy reach a new steady state after an economic reform.
  - How to compute the transition from one steady state to another.

# References

- Krueger, Mitman and Perri (2016, Handbook of Macro)*: Application of the model to the great recession.
- Boppart, Krusell and Mitman (2018, JEDC)*: Intuitive paper on how transition dynamics can be used to simulate aggregate shocks (+ history about the MIT shocks).
- Auclert, Bardóczy, Rognlie and Straub (2021, ECTA)*: State-of-the-art method to solve HA models with aggregate uncertainty.
- Krusell and Smith (1998, JPE): original paper outlining the famous algorithm.
- Heer and Maussner (2009): Ch. 8 and 10; Fehr and Kindermann (2019): Ch. 11: Textbook treatment of the computational methods.
- Algan et al (2014, Handbook of Computational Economics): Entire handbook on how to solve HA economies with aggregate uncertainty. See also their special edition on the JEDC.

# Krueger, Mitman and Perri (2016, Handbook of Macro)

- **Question:** How important is household heterogeneity for the amplification and propagation of macroeconomic shocks?
- Focus on the US Great Recession of 2007–09.
- Heterogeneity: earnings, wealth, and household preferences.
- Consequences for cross-sectional inequality in disposable income and consumption expenditures.

# Krueger, Mitman and Perri (2016, Handbook of Macro)

**Method:**

- Summarize empirical facts about the joint distribution of income, wealth, and consumption before and during the great recession.
- Compute various versions of the HA model with aggregate uncertainty and study its cross-sectional and dynamic properties.
    - Simple version of the model "replicates" the results of representative agent model.
    - Model extension with life-cycle, unemployment insurance and social security does a much better job.
- Study the impact of social insurance policies.

# Empirical Evidence: Levels

- **Data**: PSID (2004, 2006, 2008, and 2010). New version covers income, wealth, and consumption.

**Table 2** PSID Households across the net worth distribution: 2006

| | % Share of: | | | % Expend. Rate | | Head's | |
|---|---|---|---|---|---|---|---|
| NW Q | Earn. | Disp. $Y$ | Expend. | Earn. | Disp. $Y$ | Age | Edu. (yrs) |
| Q1 | 9.8 | 8.7 | 11.3 | 95.1 | 90.0 | 39.2 | 12 |
| Q2 | 12.9 | 11.2 | 12.4 | 79.3 | 76.4 | 40.3 | 12 |
| Q3 | 18.0 | 16.7 | 16.8 | 77.5 | 69.8 | 42.3 | 12.4 |
| Q4 | 22.3 | 22.1 | 22.4 | 82.3 | 69.6 | 46.2 | 12.7 |
| Q5 | 37.0 | 41.2 | 37.2 | 83.0 | 62.5 | 48.8 | 13.9 |
| | Correlation with net worth | | | | | | |
| | 0.26 | 0.42 | 0.20 | | | | |

# Empirical Evidence: Changes

**Table 3** Annualized changes in selected variables across PSID net worth

| | Net worth[a] | | | | Disp. *Y* (%) | | Cons. Exp.(%) | | Exp. Rate (pp) | |
|---|---|---|---|---|---|---|---|---|---|---|
| | (1) 04–06 | | (2) 06–10 | | (3) 04–06 | (4) 06–10 | (5) 04–06 | (6) 06–10 | (7) 04–06 | (8) 06–10 |
| **All** | **15.7** | **44.6** | **−3.0** | **−10** | **4.1** | **1.2** | **5.6** | **−1.3** | **0.9** | **−1.6** |
| NW Q | | | | | | | | | | |
| Q1 | NA | 12.9 | NA | 6.6 | 7.4 | 6.7 | 7.1 | 0.6 | −0.2 | −4.2 |
| Q2 | 121.9 | 19.5 | 24.4 | 3.7 | 6.7 | 4.1 | 7.2 | 2 | 0.3 | −1.3 |
| Q3 | 32.9 | 23.6 | 4.3 | 3.3 | 5.1 | 1.8 | 9 | 0 | 2.3 | −1.1 |
| Q4 | 17.0 | 34.7 | 1.7 | 3.8 | 5.0 | 1.7 | 5.9 | −1.5 | 0.5 | −2 |
| Q5 | 11.6 | 132.2 | −4.9 | −68.4 | 1.8 | −1.2 | 2.7 | −3.5 | 0.5 | −1.4 |

[a] The first figure is the percentage change (growth rate), the second is the change in 000's of dollars.

- **Last column**: saving rates increase relatively more for wealth-poor households during the recession.

# A Business Cycle Model with HH Heterogeneity

**Ingredients:**

- Idiosyncratic individual shocks + incomplete markets a la Aiyagari-Hugget.
- Aggregate Shocks in the spirit of the Real Business Cycle literature.
- 2 stages life cycle: young (workers) and old (retiree).
- Ex-ante heterogeneity in $\beta$.
- Government policy: unemployment insurance and social security.

# Production Technology

- Aggregate production function is Cobb-Douglas over capital and labor:

$$Y = Z^* K^\alpha N^{1-\alpha} \tag{1}$$

- $Z^* \equiv Z C^\omega$, where $\omega \geq 0$.

- **Aggregate shock**: $Z$ follows a 2-state Markov with transition matrix $\pi(Z'|Z)$:
  - $Z \in \{Z_l, Z_h\}$. $Z_l$: recession, $Z_h$ : normal times.

- **Demand externality**: $C^\omega$
  - If $\omega = 0$, standard neoclassical production function.
  - If $\omega > 0$, production is partially determined by demand.

# Households

- Standard utility over consumption $u(c)$, they cannot borrow, $a' \geq 0$.
- Ex-ante and fixed heterogeneous discount factor $\beta \in B$.
- Two idiosyncratic states:
  - Employment: $s \in \{e, u\}$. Transition matrix depends on aggregate state: $\pi(s'|s, Z', Z)$.
  - Income: $\gamma$. Transition matrix is independent of the aggregate state: $\pi(\gamma'|\gamma)$.
- Stochastic life cycle:
  - households are born as workers and with probability $1 - \theta$ they retire;
  - after retiring, they receive pensions and with probability $1 - \nu$ they die.

# Government Policy

- Pensions and unemployment benefits are financed using proportional labor taxes.
- Government runs a balance budget system every period.
- **Unemployment insurance**:
  - Pays a fraction $\rho \in [0, 1)$ of the household potential income: $b = \rho w \gamma$.
  - Financed with tax rate, $\tau(Z)$. The tax adjusts to maintain the budget balanced. In recessions, the tax rate increases.
- **Pension benefits**:
  - Financed with a fixed social security contribution $\tau_{ss}$.
  - Pension benefits, $b_{ss}$, adjust to maintain the budget balanced. In recessions, the pension decreases.

# Household Bellman Equation: Retiree

$$V_R(a,\beta;Z,\Phi) = \max_{c,\ a'\geq 0}\left\{u(c) + \nu\beta\sum_{Z'\in Z}\pi(Z'|Z)V_R(a',\beta;Z',\Phi')\right\}$$

$$\text{s.t} \qquad c + a' = b_{ss}(Z,\Phi) + (1 + r(Z,\Phi) - \delta)a/\nu$$

$$\Phi' = H(Z,\Phi,Z')$$

where $\Phi$ is the distribution of agents in the economy.

- Individual state: $(a,\beta)$, aggregate state: $(Z,\Phi)$.
- $H()$: Rational expectations function. The agents correctly forecast the next period distribution, given the state of the economy.

# Household Bellman Equation: Worker

$$V_w(s,\gamma,a,\beta;Z,\Phi) = \max_{c,\ a'\geq 0}\left\{\{u(c)+\beta\sum_{(Z',s',\gamma')\in(Z,s,\gamma)}\pi(Z'|Z)\pi(s'|s,Z',Z)\pi(\gamma'|\gamma)\right.$$

$$\left.\times\left[\theta V_W(s',\gamma',a',\beta;Z',\Phi')+(1-\theta)V_R(s',\gamma',a',\beta;Z',\Phi')\right]\right\}$$

$$\text{s.t} \qquad c+a' = (1-\tau(Z)-\tau_{ss})\gamma w(Z,\Phi)[1-(1-\rho)\mathbf{1}_{s=u}]+(1+r(Z,\Phi)-\delta)a$$

$$\Phi' = H(Z,\Phi,Z')$$

- Individual state: $(a,s,\gamma,\beta)$, aggregate state: $(Z,\Phi)$.
- Employed earnings: $\gamma w$; unemployed earnings $\rho\gamma w$.

# Equilibrium

- Prices are given by FOCs of firm's problem:

$$w(Z,\Phi) = Z\alpha \left(\frac{K(Z,\Phi)}{N(Z)}\right)^{1-\alpha} \quad \text{and} \quad r(Z,\Phi) = Z(1-\alpha)\left(\frac{N(Z)}{K(Z,\Phi)}\right)^{\alpha}$$

  where aggregate employment $N(Z)$ is given by the distributions of the Markov process (which depends on $Z$).

- Asset market clears:

$$K(Z,\Phi) = \int a d\Phi$$

- The distribution evolves according to the function: $\Phi' = H(Z,\Phi,Z')$. In equilibrium, this function is *consistent* with the individual decisions.

# Digression: Krusell & Smith (1998)

- Because prices are allowed to vary over the cycles and they are needed for the household problem: the aggregate state, $(Z, \Phi)$, is part of the state of the HH.

- **Problem:** the distribution, $\Phi$, is a high-dimensional object and the state space increases substantially.

- **Krussel & Smith (1998)**: instead of using the entire distribution, just use some moments of the distribution:
  - Households are "boundedly rational" on how the distribution evolves.
  - In this class of models, the **mean** is enough to correctly forecast prices:

$$\Phi' = H(Z, \Phi, Z') \qquad \Rightarrow \qquad K' = H(Z, K, Z') \tag{2}$$

# Digression: Krusell & Smith (1998)

- Substitute $\Phi$ by $K$. Example:

$$V_R(a,\beta;Z,K) = \max_{c,\ a'\geq 0} \left\{ u(c) + \nu\beta \sum_{Z'\in Z} \pi(Z'|Z) V_R(a',\beta;Z',K') \right\}$$

$$\text{s.t} \qquad c + a' = b_{ss}(Z,K) + (1 + r(Z,K) - \delta)a/\nu$$

$$K' = H(Z,K,Z')$$

- **Intuition:** the mean of $\Phi$ works well to forecast prices because the savings policy function is approximately linear.
- For more complex models, one may need higher moments.

# Back to Krueger, Mitman and Perri: Models

**Table 5** Taxonomy of different versions of the model used in the chapter

| Name | Discounting | Techn. | Soc. Ins. |
|---|---|---|---|
| KS | $\beta = \overline{\beta}$ | $\omega = 0$ | $\rho = 1\%$ |
| Het. $\beta$ | $\beta \in [\overline{\beta} - \epsilon \, \overline{\beta} + \epsilon]$ | $\omega = 0$ | $\rho = 50\%$ |
| Het. $\beta$ | $\beta \in [\overline{\beta} - \epsilon \, \overline{\beta} + \epsilon]$ | $\omega = 0$ | $\rho = 10\%$ |
| Dem. Ext. | $\beta \in [\overline{\beta} - \epsilon \, \overline{\beta} + \epsilon]$ | $\omega > 0$ | $\rho = 50\%$ |

- **KS**: Basic model, very similar to Krusell-Smith (1998);
- **Benchmark model**: second row, calibrated to match the US economy.

# Calibration

- Model calibrated to quarterly data. $\alpha = 0.36$, $\delta = 0.025$, $u(c) = \log(c)$.
- $Z$ duration and GDP drop of a "severe recession".
- $s$ separation and job-finding rates, $\gamma$ comes from a persistent-transitory process estimated from PSID.
- $\theta$, $\nu$: a working period of 40 years and a retirement period of 15 years.
- $\beta$ from uniform distribution: match Gini for the wealth distribution.
- Policy: unemployment benefits of 50%, $\rho = 0.5$. Pension of $40\%$ of avg. wage.

# Evaluating the Model: Wealth Distribution

**Table 6** Net worth distributions: Data vs models

| | Data | | Models | |
|---|---|---|---|---|
| **% Share held by:** | **PSID, 06** | **SCF, 07** | **Bench** | **KS** |
| Q1 | −0.9 | −0.2 | 0.3 | 6.9 |
| Q2 | 0.8 | 1.2 | 1.2 | 11.7 |
| Q3 | 4.4 | 4.6 | 4.7 | 16.0 |
| Q4 | 13.0 | 11.9 | 16.0 | 22.3 |
| Q5 | 82.7 | 82.5 | 77.8 | 43.0 |
| 90–95 | 13.7 | 11.1 | 17.9 | 10.5 |
| 95–99 | 22.8 | 25.3 | 26.0 | 11.8 |
| *T*1% | 30.9 | 33.5 | 14.2 | 5.0 |
| Gini | 0.77 | 0.78 | 0.77 | 0.35 |

- Benchmark matches the wealth distribution; but fails in the top 1%. KS fails.

# Evaluating the Model: Joint Distribution

**Table 8** Selected variables by net worth: Data vs models

| | % Share of: | | | | | | % Expend. rate | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Earnings | | Disp. $Y$ | | Expend. | | Earnings | | Disp. $Y$ | |
| NW Q | Data | Mod | Data | Mod | Data | Mod | Data | Mod | Data | Mod |
| Q1 | 9.8 | 6.5 | 8.7 | 6.0 | 11.3 | 6.6 | 95.1 | 96.5 | 90.0 | 90.4 |
| Q2 | 12.9 | 11.8 | 11.2 | 10.5 | 12.4 | 11.3 | 79.3 | 90.3 | 76.4 | 86.9 |
| Q3 | 18.0 | 18.2 | 16.7 | 16.6 | 16.8 | 16.6 | 77.5 | 86.0 | 69.8 | 81.1 |
| Q4 | 22.3 | 25.5 | 22.1 | 24.3 | 22.4 | 23.6 | 82.3 | 87.3 | 69.6 | 78.5 |
| Q5 | 37.0 | 38.0 | 41.2 | 42.7 | 37.2 | 42.0 | 83.0 | 104.5 | 62.5 | 79.6 |
| | Correlation with net worth | | | | | | | | | |
| | 0.26 | 0.46 | 0.42 | 0.67 | 0.20 | 0.76 | | | | |

- Qualitatively close to the data, but quantitative a bit far; wealth poor consuming too little, and the wealth rich consuming too much.

# Evaluating the Model: Dynamics in Normal Times

**Table 9** Annualized changes in selected variables by net worth in normal times (2004-06): Data vs model

| | Net worth (%) | | Disp. $Y$ (%) | | Expend (%) | | Exp. Rate (pp) | |
|---|---|---|---|---|---|---|---|---|
| NW Q | Data | Model | Data | Model | Data | Model | Data | Model |
| Q1 | NaN | 44 | 7.4 | 7.2 | 7.1 | 6.7 | −0.2 | −0.4 |
| Q2 | 122 | 33 | 6.7 | 3.1 | 7.2 | 3.6 | 0.3 | 0.5 |
| Q3 | 33 | 20 | 5.1 | 1.6 | 9 | 2.5 | 2.3 | 0.8 |
| Q4 | 17 | 9 | 5 | 0.5 | 5.9 | 1.7 | 0.5 | 1.2 |
| Q5 | 12 | 3 | 1.8 | −1.0 | 2.7 | 0.5 | 0.5 | 1.4 |
| All | 16 | 5 | 4.1 | 0.7 | 5.6 | 1.8 | 0.9 | 0.7 |

- Slightly too much downward and upward mobility on income, but in general good job.

# Evaluating the Model: Dynamics in Recession

**Table 10** Annualized changes in selected variables by net worth in a severe recession: Data vs model

| | Net worth (%) | | Disp. $Y$ (%) | | Expend. (%) | | Exp. rate (pp) | |
|---|---|---|---|---|---|---|---|---|
| **NW Q** | **Data** | **Model** | **Data** | **Model** | **Data** | **Model** | **Data** | **Model** |
| Q1 | NaN | 24 | 6.7 | 4.9 | 0.6 | 4.5 | −4.2 | −0.4 |
| Q2 | 24 | 15 | 4.1 | 0.3 | 2.0 | 1.2 | −1.3 | 0.8 |
| Q3 | 4 | 8 | 1.8 | −2.4 | 0.8 | 0.0 | −1.1 | 2.2 |
| Q4 | 2 | 4 | 1.7 | −4.0 | −1.7 | −1.5 | −2.0 | 3.2 |
| Q5 | −5 | −1 | −1.2 | −6.4 | −3.7 | −3.5 | −1.4 | 4.6 |
| All | −3 | 1 | 1.2 | −3.7 | −1.3 | −0.8 | −1.6 | 2.0 |

- Consumption-savings in the recession: $\downarrow$ savings because of consumption smoothing; $\uparrow$ savings because of precautionary savings.
  - In the model: the first is stronger for the richer, but the latter is stronger for the poorer.

# Aggregate Shock: Krusell-Smith vs Representative Agent

- The Krusell-Smith economy is remarkably similar to the representative agent in the aggregate.
- **Intuition**: without too many constrained agents, the HA economy behaves as a RA.

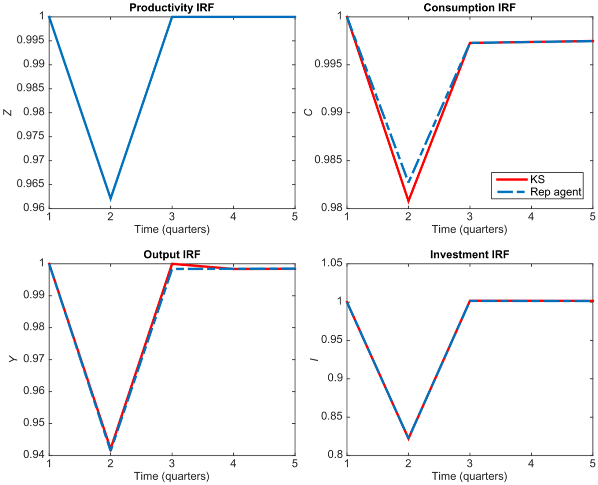
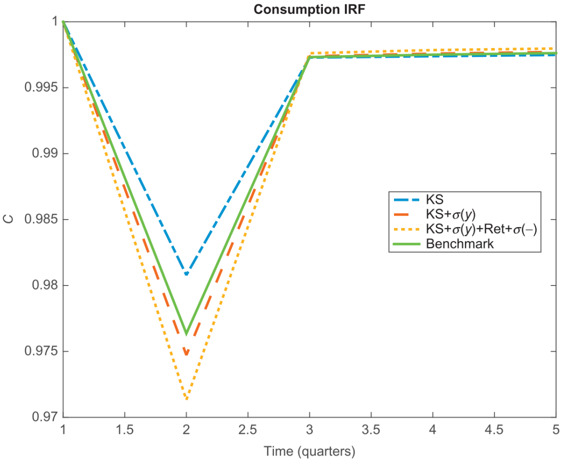



**Fig. 3** Impulse response functions (IRF) to aggregate technology shock in KS and RA economies

# Aggregate Shock: All models

- Benchmark generates a larger drop in consumption than KS economy.
- Largely accounted by income risk on top of employment risk.
- Recall that benchmark economy has high unemployment benefits.



# Differences between KS and Benchmark Economy

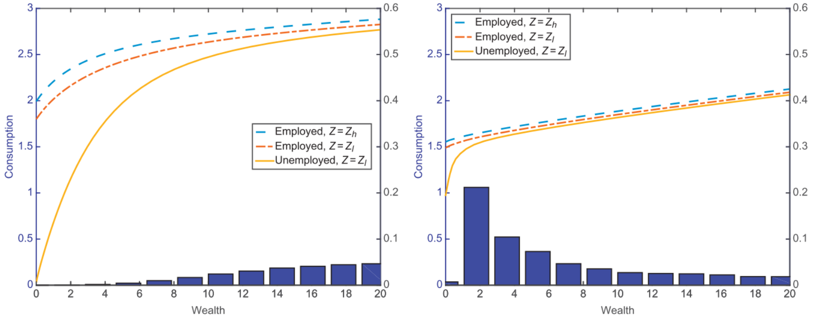



**Fig. 5** Consumption function and wealth distribution: Krusell–Smith (left panel) and benchmark (right panel).

- Benchmark generates a larger drop in consumption because it has a larger share of low wealth households.
- The low wealth consumes more in the benchmark because of unemployment benefits.

# The Role of Unemployment Insurance

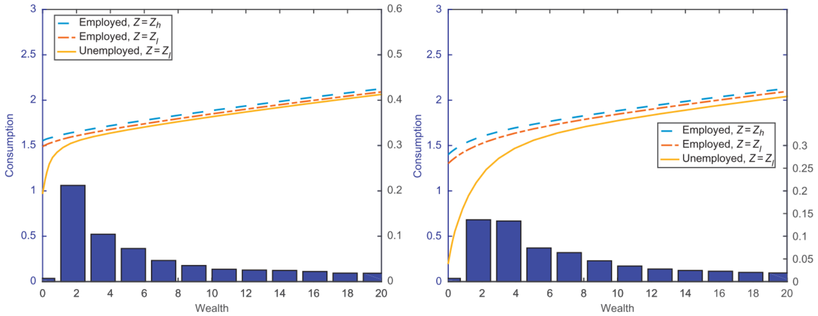



**Fig. 10** Consumption function and wealth distribution: Benchmark (left panel) and low UI (right panel).

- Unemployment benefits help the low wealth poor to consume in bad times.
- Aggregate consumption falls much more in recessions without UI.

# Demand Externality

- Keynesian flavor increases the size of the recession.
- Lots of wealth poor ⇒ large drop in consumption ⇒ demand externality ⇒ further drops output.

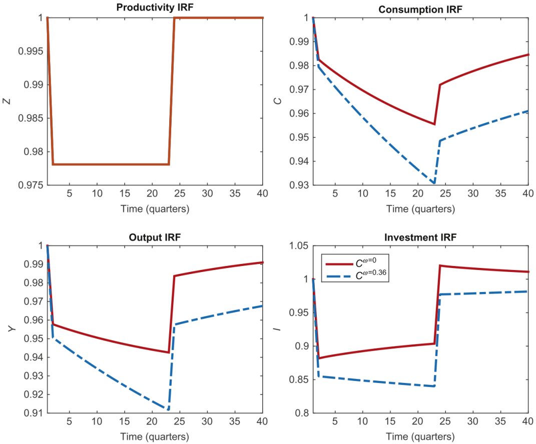



**Fig. 14** Impulse response to identical aggregate technology shock: Comparison between economies with and without demand externality.

# Conclusion: Krueger, Mitman and Perri

- Simple heterogeneity a la Krusell-Smith/Aiyagari is not enough to generate differences from the representative agent model.
- Other ingredients should be added to get a meaningful wealth distribution.
- Low wealth agents are key to getting the larger fall in consumption.
- Unemployment insurance attenuates the fall of aggregate consumption.
- Demand externality further increases the recession: motivation to include a proper microfoundation of the demand effect.

# Where to go now?

- **Income Risk and Business Cycles**: Bayer et al (2019, ECTA), McKay (2017, JME).
- **Search Frictions and Unemployment**: Ravn and Sterk (2017, JME), Nakajima (2012, JME; 2012, IER).
- **Precautionary Savings over the Cycles**: Challe and Ragot (2016, EJ), Heathcote and Perri (2017, ReStud).
- **Credit Crunch and Housing**: Guerrieri and Lorenzoni (2017, QJE), Kaplan, Mitman and Violante (2020, JPE).
- **Automatic Stabilizers**: McKay and Reis (2017, ECTA; 2021, ReStud).
- **Trends in Inequality**: Heathcote et al (2010, JPE), Heathcote, Perri and Violante (2020, RED).
- Transition dynamics of all questions we saw before.

# Computational

- To solve a heterogeneous agent economy with aggregate uncertainty the main methods are:
  - Krusell-Smith (1998, JPE) bounded rationality algorithm.
  - MIT shock (Boppart, Krusell and Mitman, 2018, JEDC).
  - Reiter (2009, JEDC) Method.
  - Auclert, Bardóczy, Rognlie and Straub (2021, ECTA) sequence space Jacobian.
- There are others/variations of algorithms. Check Algan et al (2014).

# Krusell-Smith Algorithm

- References: Krusell-Smith's original paper is easy to follow. Check also Nakajima's notes.
- Krusell-Smith: use some moments finite moments instead of the entire distribution.
  - In the model we saw before just the mean is enough: $K' = H(Z, K, Z')$.
- Approximate the function forecasting function $H()$ with a log-linear form:

$$\log K' = a_l + b_l \log K \quad \text{if } Z = Z_l$$
$$\log K' = a_h + b_h \log K \quad \text{if } Z = Z_h$$

- We have to find the parameters: $(a_l, a_h, b_l, b_h)$.

# Krusell-Smith Algorithm

Discretize the state space: $(a, s, K, Z)$. Recover the prices $r(K, Z)$ and $w(K, Z)$ for each state space using the firm's problem.

(i) Guess the parameters of the forecast function: $(a_l^0, a_h^0, b_l^0, b_h^0)$.

(ii) Given $(a_l^0, a_h^0, b_l^0, b_h^0)$, solve the Bellman Equation of the HH for all the state space $(a, s, K, Z)$.

(iii) Given the household policy functions, simulate $T$ periods:

- Draw a sequence of $Z_t$ for all $T$. Guess a initial distribution $\Phi_0$.
- Using the policy function and the sequence $Z_t$, keep updating the distribution $\Phi_t$ forward.
- Compute the mean of the distribution $K_t$ (and other moments if necessary).
- Drop the first $T_0$ periods. Now, we have a sequence $\{Z_t, K_t\}_{t=T_0}^{T}$.

# Krusell-Smith Algorithm

(iv) Using the sequence $\{Z_t, K_t\}_{t=T_0}^{T}$, run a linear regression and recover the new coefficients: $(a_l^1, a_h^1, b_l^1, b_h^1)$.

(v) Check the distance between the guess $a^0, b^0$ and the new parameters $a^1, b^1$. If it is smaller than $tol$, we are done. Otherwise, update the guess and start again:

$$a^0 = \lambda a^0 + (1-\lambda)a^1$$
$$b^0 = \lambda b^0 + (1-\lambda)b^1$$

where $\lambda \in (0,1)$ is a damping parameter.

# Krusell-Smith Algorithm: Issues

- After you finish, you must check the $R^2$ of the forecast regression. If the $R^2$ is low, you must add more moments or change the function form.
  - In Krusell-Smith, $R^2 = 0.999$, so the perceived law of motion of $K$ is very close to the actual law of motion.

- Poor initial guesses might not converge. One good guess is $a = \log K_{ss}$ and $b = 0$.

- **Good:** KS captures potential non-linearities and large shocks. For instance, asymmetries between the boom and the recession; uncertainty shocks; etc.

- **Bad:** KS can be inaccurate if there are explicitly distributional channels coming from the top of the wealth distribution. Potentially very slow.

# Reiter Method

- **Perturbation Methods**:
  - Generalization of the well-known linearization around the steady state.
  - Often used to solve representative agent models.
  - They tend to be fast, but require derivatives and some stability conditions (Blanchard-Kahn).

- Standard software (i.e., dynare) uses this method.

- Reiter (2009) propose to solve for the stationary equilibrium using global methods (projection methods), and then use perturbation methods to solve for the aggregate shock.

- If you need a refresher on Pertubation methods, check Fernandez-Villaverde's notes.

# Reiter's Method

- We can write the solution of DSGE models as a nonlinear system of difference equations:

$$E_t F(x_t, x_{t+1}, y_t, y_{t+1}) = 0 \quad (3)$$

  where $x$ is the vector of predetermined variables (state), $y$ is nonpredetermined variables (control).

- Then, we can linearize the system (either numerically or analytically) and use methods to solve the linear system of difference equations:
  - Blanchard and Kahn (1980); Uhlig (1999); Sims (2000); Rendahl (2018).

# Reiter's Method

- **Example**: Stochastic Neoclassical Growth model

$$E_t F(x_t, x_{t+1}, y_t, y_{t+1}) = E_t \begin{bmatrix} c_t^{-\gamma} - \beta E_t c_{t+1}^{-\gamma}[\alpha k_{t+1}^{\alpha-1} + 1 - \delta] \\ c_t + k_{t+1} - e^{z_t} k_t^{\alpha} - (1-\delta) k_t \\ z_{t+1} - \rho z_t - \sigma \varepsilon_{t+1} \end{bmatrix} = 0$$

  where $x = [k, z]'$ and $y = [c]$.

- First row is the Euler Equation, second is the feasibility constraint, and the last is the stochastic process of the shock.

# Reiter's Method

- **Example**: Krusell-Smith economy.

$$E_t F(x_t, x_{t+1}, y_t, y_{t+1}) = E_t \begin{bmatrix} d\Phi_{t+1} - d\Phi_t \Pi_{g_{a,t}} \\ V_t - (\overline{u}_{g_{a,t}} + \beta \Pi_{g_{a,t}} V_{t+1}) \\ z_{t+1} - \rho z_t - \sigma \varepsilon_{t+1} \\ \text{ED}(g_{a,t}, d\Phi_t, z_t, P_t) \end{bmatrix}$$

where $x = [d\Phi, z]'$ and $y = [V P]'$.

  - $d\Phi$ is the p.d.f of the distribution;
  - $P_t$ are the prices;
  - ED() is an arbitrary excess demand function (which implicitly includes firm's foc);
  - $\Pi_{g_{a,t}}$ is the transition matrix induced by the optimal policy:

$$g_{a,t} = \arg\max u(a(1 + r_t) + w_t s - a') + \beta E_t V_{t+1}(a', s', d\Phi', z')$$

# Reiter's Method

- Since we discretize both $\Phi$ and $V_t$, the first two rows must hold for ALL the idiosyncratic state.
- The number of equations that we need to linearize is exponentially large.
- Linearization is often done using numerical derivatives. Nowadays people use automatic differentiation to do the job.
- Solution (up to first order) has **certainty equivalence**: no precautionary savings because of aggregate risk.
- The method cannot capture nonlinearities or sign asymmetries.

# Transition Dynamics and MIT shocks

- Most of the time, we are interested in simulating an **impulse response function** (IRF).
- A IRF is just the deterministic transition dynamics between two steady states after an unexpected aggregate shock (a MIT shock).
- **Boppart, Krusell and Mitman (2018)** show that the IRF can be used to compute equilibrium of HA with agg. uncertainty.
- Solving for the transition dynamics is also useful if you are interested in studying the transition to a new steady state after a change in economic policy.

# MIT shock

- **MIT shock:** an unpredictable shock to the steady-state equilibrium of an economy without shocks.
  - No shocks are expected to ever materialize but nevertheless a shock now occurs!

- We can now analyze the equilibrium transition along a **perfect-foresight path** until the economy reaches the steady state.

- Some argue that **Tom Sargent** coined the term reflecting that some researchers at MIT used the method.
  - For fresh-water economists, a MIT shock is inconsistent with rational expectations!
  - "A shock of probability zero, only at MIT they can get away with that!".

# MIT shock

- Suppose a standard Ayiagari in the steady state at $t = 0$. At $t = 1$, the economy receives an (unexpected) TFP aggregate shock:

$$
\begin{aligned}
Y_t &= Z_t K_t^{\alpha} L_t^{1-\alpha} \\
\log Z_t &= \rho_z \log Z_{t-1} + \varepsilon_t
\end{aligned}
$$

  where $\varepsilon_t = 0.01$ if $t = 1$ and $\varepsilon_t = 0$ otherwise.

- If $0 < \rho_z < 1$, when $t \to \infty$, the shock vanishes and we are back to the original steady state.

- Our goal is to solve the **transition dynamics** between the two steady states.
  - Because $Z_t$ varies in the transition, aggregate variables (prices, savings, distribution) change during the transition.

# Sequential Equilibrium

- Instead of carrying the aggregate state, we index the Bellman Equation by time $t$.

$$V_t(a,s) = \max_{c,\ a' \geq 0} \left\{ u(c) + \beta \sum_{s' \in S} \pi(s'|s) V_{t+1}(a's') \right\}$$

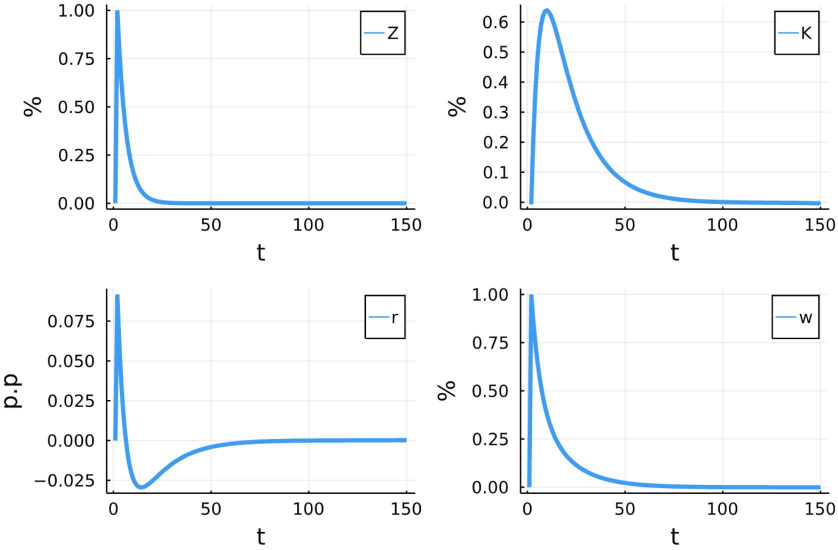
$$\text{s.t} \qquad c + a' = w_t s + (1 + r_t - \delta)a$$

- Solve for the transition means solving for the equilibrium in the asset market for all $t$:

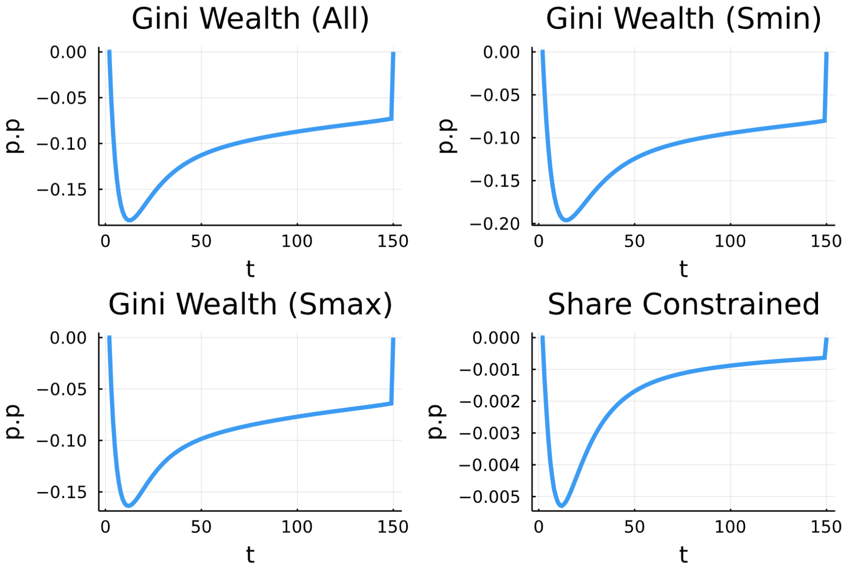
$$\int_{A \times S} a d\Phi_t(a, s; r_t) \equiv A_t(r_t) = K_t(r_t)$$

both the distribution, $\Phi_t(a, s)$, and the aggregate capital, $K_t$, are indexed by $t$.

# IRF: Standard Aiyagari Economy



# IRF: Standard Aiyagari Economy



# Transition Dynamics between Steady States

- The method is useful to compute transition between different steady states.

- **Example**: Suppose a labor tax, $\tau_l$, that is used to finance a lump-sum transfer, $T_t$. The budget constraint:

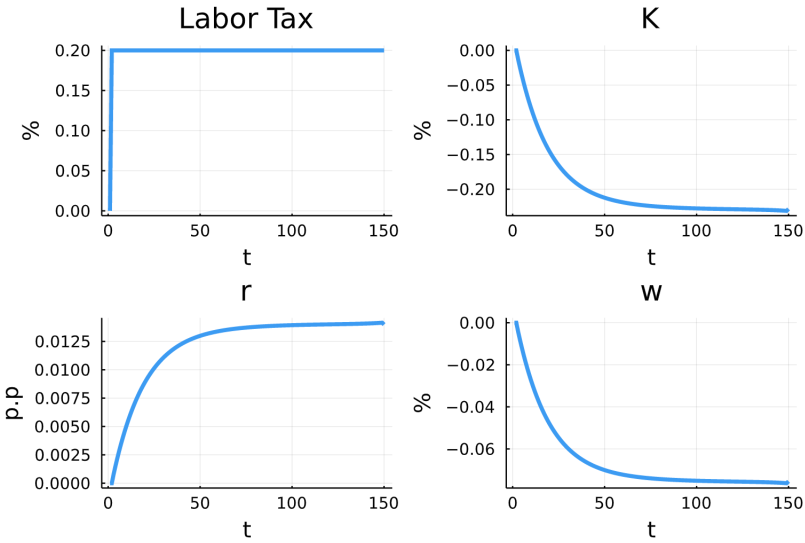
$$c + a' = w_t s(1 - \tau_l) + (1 + r_t - \delta)a + T_t.$$

  The government runs a balanced budget: $T_t = \tau_l w_t L$.

- Suppose the economy is in the SS with $\tau_l = 0$. At $t = 1$, the government decides to raise the tax rate: $\tau_l = 0.2$ (there are no aggregate shocks).

- How long does the economy take to reach the new steady state?

# Transition to New SS: Labor Tax



# Algorithm

(i) Solve for the initial and the final steady state. Select a large number of periods $T$.

(ii) Guess a path of $\{K_t^g\}_{t=2}^{T-1}$. $K_1$ and $K_T$ are given by the initial/final steady state. Recover the prices $\{r_t, w_t\}_{t=2}^{T-1}$ using the firm's problem and the sequence of $Z_t$.

(iii) Given prices, $\{r_t, w_t\}_{t=2}^{T}$, solve the value function (and policy functions) backwards from $t = T - 1, ...2$ starting from the **final steady state value function**.
- Endogenous Grid works well, but careful to use the correct prices!

(iv) Starting from the **initial steady state distribution**, simulate the distribution forward from $t = 1, ..., T - 1$ using the policy functions, $g_{a,t}(a, s)$ and the Markov process of $s$.

# Algorithm

(v) Compute aggregate savings (capital) using the distribution for all $t$: $\{K_t^s\}_{t=2}^{T-1}$.

(vi) Compute the maximum difference between the guess sequence, $\{K_t^g\}$, and the new sequence, $\{K_t^s\}$. If it is smaller than $tol$, stop. Otherwise, update the guess using the rule:

$$K_t = \lambda K_t^s + (1-\lambda)K_t^g \qquad \text{for } t = 2, .., T-1,$$

where $\lambda \in (0, 1)$ is a dampening parameter, and return to $(ii)$.

# Algorithm

- The “shooting algorithm” does not have established convergence properties but tends to work well in practice.
- The damp parameter should not be too large, otherwise, it may not converge.
- $T$ has to be large enough to allow the shock to fade out completely. Always check the last transition between times $T-1$ and $T$.
- A good initial guess is $K_{ss} = K_t$ for all $t$.
- If labor supply is endogenous you can guess $K/L$. If you need to find the eq. in other markets you have to guess an additional sequence.

# Boppart-Krusell-Mitman (2018)

- Intuitively, the method uses the impulse response function as a sufficient statistic to compute the eq. of the model.

- In theory, dynamic programming says that any aggregate statistic of the model can be computed as a function of the aggregate state: $x(Z, \Phi)$.

- Instead of using aggregate state, we can also write the aggregate stats as a function of past shocks. For example, the aggregate capital at time $t$ is:

$$K_t = K(\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ...),$$

  where $\varepsilon_t$ is the innovation of the aggregate at time $t$.

# Boppart-Krusell-Mitman (2018)

- If we assume that the model response to the shock is approximately linear, we can write $K_t$ as a linear function of past shocks:

$$K_t = \varepsilon_t K(1,0,0,...) + \varepsilon_{t-1} K(0,1,0,...) + \varepsilon_{t-2} K(0,0,1,...) + ...$$

where $K(0,1,0,...)$ is the (non-linear) response of capital at time $t$ to a shock (scaled to 1) that happened at $t-1$.

- Note that each $K$ is the response of ONLY ONE shock at each point in time.

- In the notation of BKM: $K_0 = K(1,0,0,...)$, $K_1 = K(0,1,0,...)$, etc. Then:

$$K_t = \sum_{s=0}^{\infty} \varepsilon_{t-s} K_s$$

# Boppart-Krusell-Mitman (2018)

- When we compute an impulse response function to an MIT shock, we get exactly the response of capital to a $1\%$ shock that happened $s$ periods before!

- That is, we have a sequence of $K$:

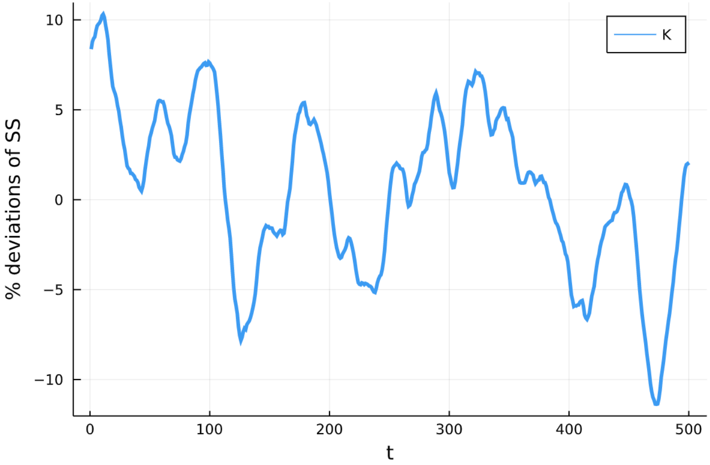
$$[K(1,0,0,...),K(0,1,0,...),K(0,0,1,...),...]$$

- In fact, we have that for all aggregate statistics of the model.

- To simulate the model, we can simply draw a sequence of shocks $\varepsilon$ and use the statistics computed by the impulse response.

# Boppart-Krusell-Mitman (2018)

Figure: Simulation of Aggregate Capital using BKM



# Boppart-Krusell-Mitman (2018)

- **Good:** It is easy to use. The only thing you need is an impulse response function. You can compute using standard dynamic programming methods.
- It is trivial to add more shocks. Because shocks are linear, you just need to simulate two IRF for each shock. Then, the final effect of the shocks is simply additive.
- **Bad:** If the model is highly non-linear or has sign-dependence it can be a poor approximation.
- As every other linear method, it assumes certainty equivalence. No second-order effects from aggregate risk; It may perform poorly if the shock brings you far from the steady state.

# State-of-the-Art Methods

- **Bayer and Luetticke (2020)**. Solving discrete time heterogeneous agent models with aggregate risk and many idiosyncratic states by perturbation.
  - Extends Reiter (2009) by applying a step that reduces the dimensionality of the model.
  - The codes are available in their website (Matlab, Python and Julia): https://www.ralphluetticke.com/.
- **Auclert, Bardóczy, Rognlie and Straub (2021)**. Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models.
  - Instead of solving for the full transition, they show that the Jacobian of the equilibrium is enough.
  - Check their lecture notes at: here.
  - Python notebooks with plenty of examples are available: here.

# Sequence-Space Jacobian

## Auclert, Bardóczy, Rognlie and Straub (2021)

- Their idea is that we can write the model in **blocks** and draw it as directed acyclic graphs (DAGs).

- A block is a part of the model that can be solved independently of the other parts. **Example**:
  - **Household Block** $\Rightarrow$ takes as given sequences of prices/policies (interest rates, wages, tax policies) and output sequences of aggregate consumption, savings, etc.

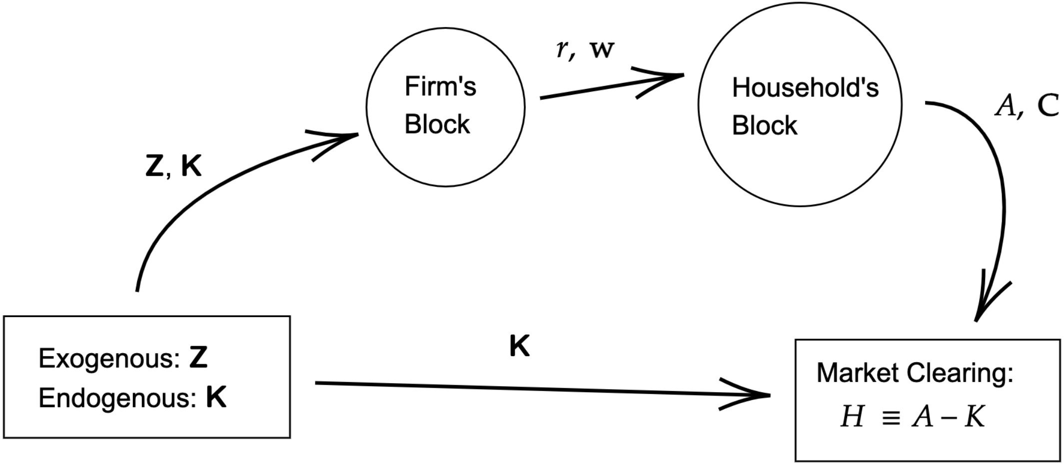
- Every block takes a sequence of inputs and outputs.

- The model is a combination of household block, firm block, government block, equilibrium block, etc.

# Sequence-Space Jacobian

- Denote sequences of variables, e.g. $Z_t$, as vectors $\mathbf{Z} = (Z_0, Z_1, ...)$.
- **Example:** Krusell-Smith Model $\rightarrow$ Exogenous: $\mathbf{Z}$, Endogenous: $\mathbf{K}$.
  - **Firm's Problem**: $\mathbf{Z}, \mathbf{K} \longrightarrow \mathbf{r}, \mathbf{w}$.
  - **Household's Problem**: $\mathbf{r}, \mathbf{w} \longrightarrow \mathbf{C}, \mathbf{A}$ (where $\mathbf{C}$ and $\mathbf{A}$ are vectors of aggregate consumption and savings, e.g., $C_t = \int g_{c,t}(a, s) d\Phi_t$).
  - **Market Clearing**: $\mathbf{A}, \mathbf{K} \longrightarrow \mathbf{H} \equiv \mathbf{A} - \mathbf{K}$ (assets mkt clearing, alternatively we could have used the goods mkt).
- **Equilibrium:** There is a sequence $\mathbf{K}$, that clears the market, $\mathbf{H} = 0$, in all periods $t$ given the sequence of exogenous variable $\mathbf{Z}$.

# Block Representation of Krusell-Smith Model



# Capital Response to Shocks

- Goal is to solve for market equilibrium given a sequence of exogenous shocks. In our example: $\mathbf{H} \equiv \mathbf{A} - \mathbf{K}$.
  - The sequence of aggregate savings, $\mathbf{A} = (A_0, A_1, ...)$, is a function of the entire sequences of interest rate, $\mathbf{r}$, and, $\mathbf{w}$. Further, $\mathbf{r}$, and wage, $\mathbf{w}$ are functions of the sequences of shock, $\mathbf{Z}$, and capital, $\mathbf{K}$.
  - Also, for every $t$, aggregate savings is a function of the **entire sequences** $\mathbf{Z}$ and $\mathbf{K}$. Then:

$$A_t(\mathbf{r}, \mathbf{w}) = A_t(\mathbf{Z}, \mathbf{K}) \tag{4}$$

- The equilibrium condition in period $t$ is:

$$H_t(\mathbf{Z}, \mathbf{K}) = A_t(\mathbf{Z}, \mathbf{K}) - K_t$$

- The sequences of equilibrium conditions are: $\mathbf{H}(\mathbf{Z}, \mathbf{K}) = \mathbf{A}(\mathbf{Z}, \mathbf{K}) - \mathbf{K}$.

# Capital Response to Shocks

- Auclert et al (2021) $\Rightarrow$ we don't need to solve for the entire equilibrium sequence to recover the response of $\mathbf{K}$ to $\mathbf{Z}$. Just need to look **Jacobians**.
- From the implicit function theorem, the linear impulse response of $\mathbf{K}$ to a transitory technology shock $d\mathbf{Z} = (dZ_0, dZ_1, ...)'$ is:

$$d\mathbf{K} = \mathbf{H}_{\mathbf{K}}^{-1}\mathbf{H}_{\mathbf{Z}}d\mathbf{Z}$$

  where $\mathbf{H}_{\mathbf{K}}$ and $\mathbf{H}_{\mathbf{Z}}$ are the Jacobians of $\mathbf{H}$ with respect to $\mathbf{K}$ and $\mathbf{Z}$, evaluated at the steady state.
- Once we have $d\mathbf{K}$, we can easily compute the response of other variables.

# The Jacobians

- To compute $\mathbf{H}_\mathbf{K}$ and $\mathbf{H}_\mathbf{Z}$, we may have to use the chain-rule.

- For example, the eq. response to $\mathbf{Z}$ is the response of $\mathbf{A}$ to changes in $\mathbf{r}$ and, $\mathbf{w}$, which further respond to $\mathbf{Z}$. We can write as a composite of Jacobians:

$$\mathbf{H}_\mathbf{Z} = \mathbf{J}^{A,r} \cdot \mathbf{J}^{r,Z} + \mathbf{J}^{A,w} \cdot \mathbf{J}^{w,Z}$$

  where $\mathbf{J}^{A,r}$ is the Jacobian of $\mathbf{A}$ to $\mathbf{r}$, and so on.

- The Jacobians of $\mathbf{H}$ are just the chain-rule of each model blocks' Jacobians ($\mathbf{J}$).

# The Jacobians

- What the Jacobians look like? Depends how complicated are model blocks.
- Some are very simple, some are complicated. The "Representative firm block" is simple.
- **Example:** $\mathbf{w}$ only depends on the contemporaneous $\mathbf{Z}$.
  - $w_t = (1-\alpha) Z_t \left(\frac{K_t}{N_t}\right)^{\alpha}$. Then, the Jacobian is:

$$\mathbf{J}^{w,Z} = \begin{bmatrix} \frac{\partial w_0}{\partial Z_0} & \frac{\partial w_0}{\partial Z_1} & \cdots & \frac{\partial w_0}{\partial Z_T} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial w_T}{\partial Z_0} & \frac{\partial w_T}{\partial Z_1} & \cdots & \frac{\partial w_T}{\partial Z_T} \end{bmatrix} = \begin{bmatrix} (1-\alpha)\left(\frac{K_0}{N_0}\right)^{\alpha} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & (1-\alpha)\left(\frac{K_T}{N_T}\right)^{\alpha} \end{bmatrix}$$

  - Note that we can exploit the sparsity of the matrix.

# The Jacobians

- The household Jacobian is complicated. Since the EE is forward looking, future shocks are anticipated by the household..
- **Example:** $\mathbf{A}$ depends on the entire path of $\mathbf{w}$.
  - Household changes its behavior in time $t$, once she understands her earnings change in time $t+s$.
  - Since $A_t$ is aggregate savings, we just need that *some* households change their behavior to change $A_t$.

$$\mathbf{J}^{A,w} = \begin{bmatrix} \frac{\partial A_0}{\partial w_0} & \frac{\partial A_0}{\partial w_1} & \cdots & \frac{\partial A_0}{\partial w_T} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial A_T}{\partial w_0} & \frac{\partial A_T}{\partial w_1} & \cdots & \frac{\partial A_T}{\partial w_T} \end{bmatrix}$$

  - Matrix is not sparse anymore.

# Sequence-Space Jacobian

- Once we have the Jacobians of each model block, we can compute the response to any type of shocks, IRF, or transition dynamics for a new SS.
- The key is to compute the Jacobians efficiently.
  - Auclert et al (2021) develops an algorithm based on "news shocks" (i.e., learning today that future income increases) ⇒ Fake News Algorithm.
  - Also must re-use the Jacobians so we only need to compute them once.
- The algorithm allows us to solve even very complex HANK models.
- It can also be applied to more general models (entry-exit, discrete choices, etc), but some details must be taken care of.
  - **Limitations** ⇒ models where the Bellman equation depends directly on the distribution (e.g., wage posting search models).