Quantitative Macroeconomics
Problem Set 0
Deadline: delivery is not required.

1. **(Lucas Span-of-control (1978))**. Consider a simplified version of Lucas (1978) span-of-control model. Every period, agents decide whether operate as an entrepreneur or work for the market wage, w. In case the agent decides to be an entrepreneur, she runs a business and hires workers to produce the final good using the following production function:

$$y = zn^{\alpha} \qquad \alpha \in (0, 1)$$

where z is the managerial ability and n is the number of workers hired by the entrepreneur.

Agents are heterogeneous in their managerial ability z. When they are born, they draw z from a distribution $F(z)$ with support $[z_{min}, \infty)$.

Every period, agents maximize their income. There are no savings so the full specification of the utility function is inconsequential and the problem is static.

   (a) Derive the labor demand and the associated profit of an individual entrepreneur as a function of z and w.

   (b) Let z^* be the cutoff so the agents $z^* \geq z$ decide to operate as an entrepreneur and $z^* < z$ work for the market wage. Use the indifference condition and find w as a function of z^* and parameters.

   (c) Let $F(z)$ be a Pareto distribution:

$$F(z) = 1 - \left(\frac{z_{min}}{z}\right)^{\gamma},$$
$$f(z) = \frac{\gamma z_{min}^{\gamma}}{z^{\gamma+1}}.$$

   Suppose $\gamma(1 - \alpha) - 1 > 0$. Use the labor market clearing condition and w to show that

$$z^* = z_{min} \left(\frac{\gamma - 1}{\gamma(1 - \alpha) - 1}\right)^{\frac{1}{\gamma}}.$$

   (d) We will now solve the model numerically. Suppose $\alpha = 0.4$, $z_{min} = 1$ and $\gamma = 3$. Using the individual labor demand and the p.d.f $f(z)$, write a function that computes the aggregate labor demand as a function of w (alternatively, you can write as a function of z^*). Note that you will have to compute an integral numerically. I recommend you to use quadrature methods (such as Gaussian quadrature). The aggregate labor demand reads:

$$N^d(w) = \int_{z^*(w)}^{\infty} n^d(z, w) f(z) dz,$$

   where $z^*(w)$ is the indifference cutoff, as a function of w, derived in part (b).

(e) Write a function that computes the excess labor demand (i.e., aggregate demand minus aggregate supply) as a function of w (or z^*):

$$\Phi(w) = N^d(w) - F(z^*(w)).$$

Use a root-finding routine to compute the equilibrium wage, w, and the cutoff z^*. Make sure you find the same value as given by the closed-form expression of part (c).

(f) Compute some statistics: income inequality (Gini or variance), average firm size, the share of workers, etc. How do these changes if we increase α?

2. **(Solving the Neoclassical Growth Model using VFI)**. Consider the standard Neoclassical Growth Model in infinite horizon. The production function is given by k_t^α, the capital law of motion is:

$$k_{t+1} = k_t(1-\delta) + k_t^\alpha - c_t.$$

The representative household chooses a consumption sequence to maximize the following utility function:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t).$$

The Bellman equation of the problem is:

$$V(k) = \max_{k'}\{\log(k^\alpha + k(1-\delta) - k') + \beta V(k')\}$$

and the associated policy function: $k' = g_t(k)$.

(a) Describe the value function iteration algorithm to find the value and policy function.

(b) Consider $\alpha = 0.3$, $\beta = 0.96$ e $\delta = 0.1$. Discretize the capital state space in $n_k = 500$ equidistant points, with $k_{min} = 2k_{ss}/n_k$ e $k_{max} = 2k_{ss}$ (k_{ss} is the capital in the steady state). Implement the algorithm in a programming language of your choice.

(c) Plot the policy function in a figure.

(d) Let $k_0 = k_{min}$. Use the policy function to simulate the optimal capital sequence. Plot the optimal path until it reaches the steady state.