Macroeconomics I

Neoclassical Growth Model

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What We Learn in This Chapter

- How to solve the Neoclassical growth model using the social planner and the descentralized competitive equilibrium.

- How to use the Hamiltonian to solve continuous time problems.

- How to use the phase diagram to learn about the transitional dynamics of the neoclassical growth model.

- What are the conditions that guarantee a balanced growth path.

References

- PhD Macrobook Ch. 4.
- Acemoglu Ch. 7 and 8.
- Dirk Krueger Ch. 3 and 9.

Introduction

- We are going to build our first dynamic GE macroeconomic model.
- In particular, we will study the **Neoclassical growth model** (AKA Ramsey-Cass-Koopmans).
- In this model, savings rate are endogenous and respond to changes in the environment.
- The model is the building block for most of the more complex models out there.

Discrete Time: Neoclassical Growth Model

- **Environment**: No uncertainty; Single good that can be consumed or invested $Y_t = C_t + I_t$. No population growth (yet), $L_t = 1$. No TFP growth (yet).
- **Preferences:** $U(\{C_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(C_t)$
- **Technology**: $Y_t = F(K_t, L_t) = F(k_t, 1) \equiv f(k_t)$ and $I_t = K_{t+1} - (1-\delta)K_t$.
  - Production function follows the same assumptions as in Solow.
- **Government:** None (yet).
- **Endowments**: Initial capital K_0.
- **Equilibrium concept:** Competitive.

Solution: Sequences $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ (everything now will be in per-capita terms).

Utility

- The utility function will follow the usual assumptions:
  - Time-separable, $u'(c) > 0$, $u''(c) < 0$, $\beta \in (0, 1)$, time-invariant, and satisfies the Inada conditions.
- In particular, we will assume that the model is populated by a **representative household**.
- Solving the individual problem is enough to get the aggregate consumption, aggregate savings, etc.
- A lot of macro models assume a *representative household*. What does this mean?

Digression: Representative Agent

- Suppose that instead of a representative household, there is a continuum of households h represented by the interval $[0, 1]$.
  - Advantage of using unit population: aggregate value = average.
  - In general, $u()$ and β can depend on family h
  - Households can also be heterogeneous in their endowments: income, wealth...

- We are interested in studying aggregate variables ⇒ eventually we have to aggregate the decisions of all individuals in the economy.

- That is, the aggregate household demand, C_t, is defined as the sum of all hh in the economy:

$$C_t = \int_0^1 c_t^h dh \tag{1}$$

where c_t^h is the optimal consumption of agent h.

Digression: Representative Agent

- **Problem:** Aggregating heterogeneous agents can be complicated.
- It implies solving the decision of each individual agent individually.
- **Solution:** Assume the existence of a representative agent.
  - The aggregate demand of the economy can be represented by a representative agent making decisions subject to the aggregate budget constraint.
  - When can we do this? What do we lose?
- **Trivial solution:** Assume that preferences and endowments are equal for all h:
  - $u^h() = u(), \quad \beta^h = \beta$ and equal endowments $\Rightarrow c^h = c$.
- We don't always need to assume that all agents are equal for our model to be represented by a representative household.

Gorman Aggregation Theorem

Theorem (Gorman Aggregation Theorem)

Consider an economy with $N < \infty$ goods and a set H of agents with wealth w^h. Suppose that the preferences of each household $h \in H$ are represented by the indirect utility

$$v^h(p, w^h) = a^h(p) + b(p)w^h, \tag{2}$$

then preferences can be aggregated and represented by an agent with indirect utility

$$v(p, w) = a(p) + b(p)w, \tag{3}$$

where $a(p) = \int_{h \in H} a^h(p)dh$ and $w = \int_{h \in H} w^h dh$.

- **Proof:** Use Roy's identity to find individual demand and take the integral over h.

Gorman Aggregation Theorem

- If preferences lead to linear indirect utilities in wealth with the same $b(p)$ for all agents, we can represent individual demand for any arbitrary good:

$$c^h(p, w^h) = \alpha^h(p) + \kappa(p)w^h \tag{4}$$

- **Linear** relationship between demand and wealth!
- **Intuition**:
  - If all agents have the same marginal propensity to consume, aggregate demand only depends on aggregate wealth!
  - When reallocating wealth from one agent to another, aggregate demand does not change.

A Simple Example

- Suppose 2 agents with Cobb-Douglas utility $U(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$.
  - Capitalist receives profit $y^c = \pi$.
  - Worker receives wage $y^w = w$.
  - Aggregate income $Y = w + \pi$.

- Individual demands: $x_1^i = \alpha y^i / p_1$ and $x_2^i = (1-\alpha) y^i / p_2$ for $i = c, w$.

- Indirect utility:

$$v^i(p, y^i) = x_1^\alpha x_2^{1-\alpha} = \left(\alpha \frac{y^i}{p_1}\right)^\alpha \left((1-\alpha)\frac{y^i}{p_2}\right)^{1-\alpha} = \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} y^i$$

- Indirect utility of the representative agent with income Y:

$$v(p, Y) = \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} Y$$

Utility Function

Other examples satisfying Gorman aggregation

(i) Quasi-homothetic utilities:

$$u(x_1^h, ..., x_N^h) = \left[\sum_{j=1}^{N} (x_j - \xi_j^h)^{(\sigma-1)/\sigma} \right]^{\sigma/(\sigma-1)} \tag{5}$$

define $\tilde{x}_j^h = x_j - \xi$. As long as the solution is interior, utility admits a representative agent with $\xi_j \equiv \int_h \xi_j^h dh$.

(ii) Quasi-linear utilities

$$u(c, l) = u(c) + \phi l \tag{6}$$

Representative Agent

- There are versions of Gorman's Aggregation Theorem for dynamic economies.
- We assume u that allows for representative agents!
- From now on, we will represent households with a single representative household: $U(c_t^h) = U(c_t)$ (unless noted).
- **Exercise**: Find the representative agent of the economy with two agents from the previous section.

Solving the Model

- We already have assumptions about preferences and technology.
- **Ultimate goal:** Find equilibrium allocations (and prices). How to do it:
  (i) **Decentralized equilibrium**: find the price that equates the supply of capital/consumption with its demand.
  (ii) **Social planner**: Solve the problem of the **benevolent central planner** $\Rightarrow$ Also the optimal/efficient solution of the model.
- Given certain assumptions, the solutions to both problems are the same.
- **Social Planner's Problem**:
  - Maximizes utility given the technological and resource constraints of the economy (not subject to consumer budget constraint - but rather to the TOTAL resources of the economy).

Social Planner's Problem

- Planner chooses the allocation $\{k_{t+1}, c_t\}_{t=0}^{\infty}$ that maximizes the representative household's utility.

$$\max_{\{k_{t+1}\geq 0,\quad c_t \geq 0\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{7}$$

$$s.t. \quad c_t + i_t \leq y_t = f(k_t) \quad \forall t; \tag{8}$$

$$k_{t+1} = i_t + (1-\delta)k_t \quad \forall t; \tag{9}$$

$$k_0 > 0 \text{ given}; \tag{10}$$

- Substitute the capital accumulation equation into i_t, so the resource constraint reads: $c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$.

Solving a Dynamic Problem

- We will first solve it with a finite T using constraint optimization methods (Kuhn-Tucker).
- The Kuhn-Tucker conditions are sufficient if the objective function is concave and the constraints are convex.
- The assumptions we made about u and f guarantee that these conditions are satisfied:
    - $u(c_t)$ is increasing so the resource constraint remains with equality.
    - $u(c_t)$ is concave, so the sum of $u(c_t)$ is also concave.
    - The constraint is convex: $0 \leq k_t \leq f(k_t) + (1-\delta)k_t$.
    - Inada conditions ensure an interior solution $c > 0$ and $k > 0$.
    - Except for the last period T where $k_{T+1} = 0$.

Neoclassical Growth Model

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \left[\beta^t u(c_t) + \lambda_t \left(f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right) + \mu_t k_{t+1} \right] \tag{11}$$

- Kuhn-Tucker conditions (for all t):
  - $k_{t+1} \geq 0$, $\lambda_t \geq 0$ and $\mu_t \geq 0$.
  - Complementary slackness: $k_{t+1}\mu_t = 0$

First-order conditions...

- Note that $k_{t+1} > 0$ and $\mu_t = 0$, for all $t = 0, .., T-1$:

$$u'(c_t)\beta^t = \lambda_t \quad \text{and} \quad \lambda_t = [f'(k_{t+1}) + 1 - \delta]\lambda_{t+1} \quad t = 0, ..., T-1 \tag{12}$$

- And we find the *Euler Equation*:

$$u'(c_t) = (f'(k_{t+1}) + 1 - \delta)\beta u'(c_{t+1}) \quad t = 0, 1, ..., T-1 \tag{13}$$

Euler Equation

- The Euler Equation the trade-off between consumption and saving (or in the case of the planner allocating one unit in c_t or in i_t).

$$\underbrace{u'(c_t)}_{\text{mg. cost of invest}} = \underbrace{(f'(k_{t+1}) + 1 - \delta)}_{\text{return on investment}} \underbrace{\beta u'(c_{t+1})}_{\text{mg. utility of consuming more in } t+1} \tag{14}$$

- Marginal cost of foregoing one unit of the final good in t is equal to the discounted marginal benefit of consuming $f'(k_{t+1} + 1 - \delta$ units of the final good in $t+1$.

- Strict concavity in the utility function implies households would like to smooth consumption over the lifetime.

- Note that extra saving changes future returns via $f'(k_{t+1}) + 1 - \delta$.

Solving the Problem: Finite Time

- Note that the Euler equation is only valid until period $T-1$. The FOCs in period T:

$$u'(c_T)\beta^T = \lambda_T \quad \text{and} \quad \lambda_T = \mu_T \quad t = 0, ..., T-1 \tag{15}$$

- This implies that $\mu_T = \lambda_T > 0$ and, since $k_{T+1}\mu_T = 0$, $k_{T+1} = 0$.

- Intuitive result, since it makes no sense to take capital to $T+1$.

Finite Time Solution

- Utility-maximizing sequences must satisfy the system of difference equations (for $t = 0, ..., T-1$):

$$u'(c_t) = (f'(k_{t+1}) + 1 - \delta)\beta u'(c_{t+1}) \qquad \text{(Euler Equation)} \qquad (16)$$

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \qquad \text{(Resource Constraint)} \qquad (17)$$

- Alternatively, we can substitute c_t and write the problem as a second-order difference equation.
- Two first-order difference equations require two conditions: an initial and a terminal condition.
- These conditions are: given by k_0 and $k_{T+1} = 0$.

Infinite Time

- In finite time $k_{T+1} = 0$. But in infinite time what is the terminal condition that ensures the system of equations has a unique solution? The *Transversality Condition* (TVC).
- Note that in finite time: $\lambda_T k_{T+1} = 0$.
- The **Transversality Condition**:

$$\lim_{T \to \infty} \lambda_T k_{T+1} = \lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} = 0 \tag{18}$$

- Intuitively, it says that the shadow value of capital converges to zero (not the capital stock).
  - In this sense, it is not optimal for the planner to choose a sequence of capital involving a positive shadow value in present value (as it was not optimal to $k_{T+1} > 0$ in finite time).
- Without TVC, it is possible to find infinite sequences of c_t and k_{t+1} that satisfy the EE.
- Check the **Proof** for sufficiency of the TVC + EE in the PhD macrobook (Proposition 4.4).

Infinite Time

- With the two first-order difference equations (EE and resource constraint) and the initial and terminal condition (k_0 and TVC), we can find the optimal allocations that solve the central planner's problem.
- In most applications it is not possible to solve the problem analytically. You must:
  - Use linear approximations.
  - Solve the problem on the computer (using dynamic programming).
- There is one case you can solve the model analytically:
- **Example**: Suppose $u(c) = \log(c)$, $\delta = 1$, $f(k) = k^\alpha$ (i.e., $F()$ is Cobb-Douglas). Solve for the optimal policy, i.e. k_{t+1} as a function of k_t and the parameters.

Neoclassical Growth: Decentralized Equilibrium

- We already solved for optimal growth allocations $\Rightarrow$ Social Planner.
- We know that by the Welfare Theorems the allocations chosen by the planner are the same as the one of the competitive equilibrium (under certain conditions).
- Okay, but what about prices? And what if the Welfare Theorems don't apply?

Neoclassical Growth Model: Firms

- The representative firm's problem is the same as in the Solow Model:

$$\max_{K_t, L_t} \pi_t = F(K_t, L_t) - r_t K_t - w_t L_t \tag{19}$$

- Given the assumptions on $F(K_t, L_t)$, the FOCs are necessary and sufficient:

$$r = MPK = F_K(K, L) = f'(k) \tag{20}$$

$$w = MPL = F_L(K, L) = f(k) - f'(k)k \tag{21}$$

Neoclassical Growth Model

Households' Problem

- Households own the capital and rent it to the firms through an asset market (receiving net returns $r_t - \delta$).
- Households supply labor (inelastically) to firms.

$$\max_{\{a_{t+1},\, c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{22}$$

$$s.t. \quad c_t + a_{t+1} \leq (1 + r_t - \delta)a_t + w_t \quad \forall t; \tag{23}$$

$$k_t \geq 0 \; \forall t \quad \text{and} \quad a_0 > 0 \text{ given}; \tag{24}$$

- Note that factor prices have a t-subscript. The budget constraint is already in per-capita terms (otherwise, labor income would be $w_t L_t$).

Neoclassical Growth Model

Households' Problem

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda_t \left(w_t + (1 + r_t - \delta)a_t - a_{t+1} - c_t \right) \tag{25}$$

- Given the assumptions we made on F and u, we know that the solution will be interior and the budget constraint holds with equality.
- FOCs: $\beta^t u'(c_t) = \lambda_t$ and $\lambda_{t+1}(1 + r_{t+1} - \delta) = \lambda_t$ for all t.
- Solution of the problem is the sequence that satisfies the Euler Equation:

$$u'(c_t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1}) \quad \forall t \tag{26}$$

together with the Transversality Condition.

Households' Problem: TVC vs No-Ponzi Game

- Note that the TVC is similar to the *no-Ponzi game*, as both prevent optimal trajectories from "blowing up".
- Assuming a *no-Ponzi game* condition with equality and $a_t = k_t$, we have via TVC:

$$\lim_{T\to\infty} \lambda_T k_{T+1} = 0 \quad \text{and} \quad \lambda_t = \frac{\lambda_{t-1}}{(1 + r_t - \delta)} \tag{27}$$

- Iterating: $\lambda_T = \frac{\lambda_0}{\Pi_{t=0}^{T}(1+r_t-\delta)}$ and substituting, we arrive at no-Ponzi.
- Although they have the same utility, conceptually they are different things:
  - The *no-Ponzi game* is a restriction in the households' problem that prevents the accumulation of debt.
  - In the basic neoclassical growth model, the no-Ponzi condition is usually omitted since $a_t = k_t > 0 \; \forall t$.
  - But in more sophisticated versions (with different types of bonds, government, etc.) it may be necessary.

TVC vs No-Ponzi Game

- TVC determines the optimal choice given a set of possible sequences.
- It is a necessary and sufficient condition for the solution of the problem in the **sequential formulation** of the growth model.
  - In other words, it is a **terminal condition**.
- **Kamihigashi (2008)**: "A no-Ponzi-game condition is a constraint that prevents overaccumulation of debt, while a typical transversality condition is an optimality condition that rules out overaccumulation of wealth. They place opposite restrictions, and should not be confused."
  - They have opposite inequality signs!!!

Neoclassical Growth Model: Equilibrium Conditions

- Market clearing for capital and labor:

$$L_t^d = 1 \quad \text{and} \quad k_t^d = a_t \quad \forall t \tag{28}$$

- Market clearing in the goods market (resource constraint):

$$y_t = c_t + i_t \quad \forall t \tag{29}$$

  which is trivially satisfied by the households' budget constraint: $y_t = f(k_t) = r_t k_t + w_t$ and $i_t = k_{t+1} - (1-\delta)k_t$.

- By Walras's Law with two markets in equilibrium, the third will also be. Note that we solve for two prices for every t: r_t and w_t (the price of the final good was normalized $p_t = 1$).

Neoclassical Growth Model: Competitive Equilibrium

Definition. A competitive equilibrium is a sequence of allocations $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ for the consumer and the firm $\{k_t^d, L_t^d\}_{t=0}^{\infty}$, and prices $\{w_t, r_t\}_{t=0}^{\infty}$ such that:

1. Given k_0 and the sequence of interest rates and wages $\{r_t, w_t\}_{t=0}^{\infty}$, $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ is the solution to the household's problem.
2. Given the sequence of interest rates and wages $\{r_t, w_t\}_{t=0}^{\infty}$, $\{k_t^d, L_t^d\}_{t=0}^{\infty}$ is the solution to the firm's problem.
3. The markets clear for every t:

$$
\begin{aligned}
L_t^d &= 1 \\
k_t^d &= a_t \\
f(k_t) &= c_t + k_{t+1} - (1-\delta)k_t
\end{aligned}
$$

Equilibrium in Neoclassical Growth

- Combining the household's solution (Euler equation + budget constraint) with the firm's solution (factor market price equal to marginal product), we have:

$$
\begin{aligned}
u'(c_t) &= \beta(1 + r_{t+1} - \delta)u'(c_{t+1}) \quad \forall t \\
c_t + k_{t+1} - (1-\delta)k_t &= r_t k_t + w_t = y_t = f(k_t) \quad \forall t \\
r_t &= F_K(K_t, L_t) = MPK \quad \forall t \\
w_t &= F_L(K_t, L_t) = MPL \quad \forall t
\end{aligned}
$$

- Resulting in the same system as the social planner's:

$$
\begin{aligned}
u'(c_t) &= (f'(k_{t+1} + 1 - \delta)\beta u'(c_{t+1}) && \text{(Euler Equation)} \\
c_t + k_{t+1} &= f(k_t) + (1-\delta)k_t && \text{(Resource Constraint)}
\end{aligned}
$$

- EE + resource constraint + k_0 + TVC characterize the equilibrium sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.

Steady State

- **Steady State**: An economy is in a steady state when its variables assume a constant value over time.

$$k_{ss} = k_{t+1} = k_t$$
$$c_{ss} = c_{t+1} = c_t.$$

- Given our assumptions - especially concavity of F and constant returns to scale - the economy will converge to a steady states (intuition comes from the Solow model).

Steady State

- For the economy to reach the steady state, it suffices to start with $k_0 > 0$.
- Note that using the EE: $u'(c_{ss}) = \beta(1 + r_{ss} - \delta)u'(c_{ss})$ together with $r_{ss} = f'(k_{ss})$ we can easily find k_{ss}.
  - If $k_0 < k_{ss}$, the economy will accumulate capital until it reaches the steady state.
  - If $k_0 > k_{ss}$, the economy will decumulate capital until it reaches the steady state.
- We will study the accumulation dynamics in more detail later.
- **Example:** Find k_{ss} given $f(k) = k^\alpha$.

Steady State

Steady State with $F(K,L) = K^\alpha L^{1-\alpha}$

$$u'(c_{ss}) = \beta(1 + r_{ss} - \delta)u'(c_{ss}) \tag{30}$$

$$c_{ss} + \delta k_{ss} = r_{ss}k_{ss} + w_{ss}L_{ss} \tag{31}$$

$$r_{ss} = \alpha\left(\frac{k_{ss}}{L_{ss}}\right)^{\alpha-1} \tag{32}$$

$$w_{ss} = (1-\alpha)\left(\frac{k_{ss}}{L_{ss}}\right)^{\alpha} \tag{33}$$

- Given that $L_{ss} = 1$, it is a system of 4 equations and 4 endogenous variables $\{k_{ss}, c_{ss}, r_{ss}, w_{ss}\}$.
- There is no dynamics, so it is possible to find the analytical solution for the endogenous variables.

Dynamic Optimization in Continuous Time

Introduction

- Discrete time:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Continuous time:

$$\int_0^{\infty} e^{-\rho t} u(c_t) dt$$

- Why is discounting exponential?
- What's the difference?
  - There is no substantial difference. Some problems are naturally written in discrete time, others in continuous time (e.g., optimal stopping time problems).
  - Mathematics tends to be more elegant but sometimes more complicated.
- How to solve the problem?

Intuition

- The discount rate between discrete and continuous time are equivalent: $\beta^t = \left(\frac{1}{1+\rho}\right)^t$.
- **Intuition:** Suppose a period of t years. We can calculate compound interest:

$$\left(\frac{1}{1+r/n}\right)^t \times \cdots \times \left(\frac{1}{1+r/n}\right)^t = \left(\frac{1}{1+r/n}\right)^{nt}$$

  - If $n = 1$, we use annual interest. If $n = 4$, quarterly interest, etc.

- In continuous time, $n \to \infty$:

$$\lim_{n\to\infty} \left(\frac{1}{1+r/n}\right)^{nt} = e^{-rt}$$

- **Proof:** define $s \equiv n/r$, take the limit $s \to \infty$, and use L'Hôpital's rule.

Some Important Tricks

Continuous Time Growth:

- Growth rate over an interval Δt:

$$\frac{x_{t+\Delta t} - x_t}{x_t \Delta t} \quad \text{or in logs} \quad \frac{\ln x_{t+\Delta t} - \ln x_t}{\Delta t}$$

  taking the limit:

$$\lim_{\Delta t \to 0} \frac{x_{t+\Delta t} - x_t}{x_t \Delta t} = \frac{\dot{x}_t}{x_t} = g$$

- Suppose a variable $x = X/L$, where X grows at rate g and L at rate n:

$$\frac{\dot{x}_t}{x_t} = \frac{\dot{X}_t}{X_t} - \frac{\dot{L}_t}{L_t} = g - n.$$

Example: Consumption and Saving

- Finite continuous time: $t \in [0, T]$.

$$\max_{c_t} \int_0^T e^{-\rho t} u(c_t) dt + e^{-\rho T} V_T(a_T)$$
$$s.t. \quad \frac{\partial a_t}{\partial t} = \dot{a}_t = r a_t + w - c_t,$$
$$a_0 \text{ given and } a_T \geq 0.$$

where $V_T(a_T)$ is a terminal value (exogenous).

- Alternatively, we can impose a condition: $a_T = 0$ (not an agent's choice, but a restriction in the problem!).
- The solutions c_t and a_t are functions of time: $c : [0, T] \to \mathbb{R}$.

Example: Consumption and Saving

- How to find the budget constraint in continuous time?
- Budget constraint for a period Δt:

$$a_{t+\Delta t} = (a_t + w\Delta t - c_t\Delta t)(1 + r\Delta t)$$

- Flow vs stock: a_t is stock and $c_t,\ w$, and r are flow variables.
- Rewriting and taking the limit $\Delta t \to 0$:

$$\frac{a_{t+\Delta t} - a_t}{\Delta t} = ra_t + w - c_t + (w - c_t)r\Delta t$$

- Note, if $a_{t+\Delta t} = a_t(1 + r\Delta t) + w\Delta t - c_t\Delta t$, solution will be the same (why?).

Continuous Time

- How to solve the problem?
  - Variational calculus (we won't see it).
  - Pontryagin's Maximum Principle (analogous to the Lagrangian).
  - Hamilton-Jacobi-Bellman equation (analogous to the Bellman equation, likely won't see it).
- The approach here will be more "intuitive" and less formal. We will skip most of the theorems and proofs.
- Intuitively, much of what we've seen for discrete time has an equivalent for continuous time (transversality condition, etc).
- Acemoglu's Chapter 7 is the reference if you are interested in more details.

Pontryagin's Maximum Principle

- Consider the general problem:

$$\max_{y_t, x_t} \int_0^T f(x_t, y_t, t)dt + M(x_T)$$
$$s.t. \quad \dot{x}_t = g(x_t, y_t, t)$$
$$x_0 \text{ given.}$$

- x_t is the state vector.
- y_t is the control vector.
- f is the return function (with implicit discounting).
- g is the state's law of motion.

Pontryagin's Maximum Principle

- Define the Hamiltonian as:

$$H(x_t, y_t, \lambda_t, t) = f(x_t, y_t, t) + \lambda_t g(x_t, y_t, t)$$

where λ_t is the costate, which is of the same dimension as x_t, and is a function of time.

- **Pontryagin's Maximum Principle.** Suppose f and g are continuously differentiable and (x_t^*, y_t^*) are continuous interior solutions. Then, there exists a function λ_t^* that satisfies the necessary conditions:

$$H_x(x_t, y_t, \lambda_t, t) = -\dot{\lambda}_t^* \quad \forall t, \tag{34}$$
$$H_y(x_t, y_t, \lambda_t, t) = 0 \quad \forall t, \tag{35}$$
$$H_\lambda(x_t, y_t, \lambda_t, t) = \dot{x}_t \quad \forall t. \tag{36}$$

With the terminal condition $\lambda_T = M_x(x_t)$ (if the terminal condition is $a_T = 0$, then $\lambda_T = 0$).

Pontryagin's Maximum Principle

- For some intuition, imagine the following Lagrangian:

$$\mathcal{L}(x_t, y_t, \lambda_t, t) = \int_0^T [\underbrace{f(x_t, y_t, t) + \lambda_t g(x_t, y_t, t)}_{H(x_t, y_t, \lambda_t, t)} - \lambda_t \dot{x}_t] + M(x_T)$$

- λ_t acts as the "multiplier" and informs about the value of relaxing the constraints.
- The necessary conditions are analogous to the first-order conditions of the Lagrangian plus a "temporal' condition.

Pontryagin's Maximum Principle

- When the return function is exponentially discounted by $e^{-\rho t}$ (practically all economics' problem), it is convenient to redefine the Hamiltonian in **current value** $\hat{H}(x_t, y_t, \mu_t, t)$:

$$
\begin{aligned}
H(x_t, y_t, \lambda_t, t) =& f(x_t, y_t, t) + \lambda_t g(x_t, y_t, t) \\
H(x_t, y_t, \lambda_t, t) =& e^{-\rho t} \underbrace{(\hat{f}(x_t, y_t, t) + \mu_t g(x_t, y_t, t))}_{\hat{H}(x_t, y_t, \mu_t, t)}
\end{aligned}
$$

- Where the multiplier and the return function are in current values:

$$\mu_t = \lambda_t e^{\rho t}, \text{ and } \hat{f}(x_t, y_t, t) = f(x_t, y_t, t) e^{\rho t}.$$

- The only condition that changes is (34):

$$
\begin{aligned}
\hat{H}_x(x_t, y_t, \mu_t, t) &= \rho \mu_t - \dot{\mu}_t^* \\
\hat{H}_y(x_t, y_t, \mu_t, t) &= 0 \\
\hat{H}_\lambda(x_t, y_t, \mu_t, t) &= \dot{x}_t.
\end{aligned}
$$

Example: Consumption and Saving

- In the Consumption and Saving problem:

$$\hat{H} = u(c_t) + \mu_t(ra_t + w - c_t)$$

- Hence:

$$r\mu_t = \rho\mu_t - \dot{\mu}_t^* \quad \forall t, \tag{37}$$
$$u'(c_t) = \mu_t \quad \forall t. \tag{38}$$

- Taking the derivative with respect to time:

$$\dot{\mu}_t = u''(c_t)\dot{c}_t \tag{39}$$

- Substituting, we find the Euler's equation (in continuous time):

$$\frac{u''(c_t)\dot{c}_t}{u'(c_t)} = -(r - \rho) \tag{40}$$

Example: Consumption and Saving

- The solution is a system of ordinary differential equations:

$$\frac{u''(c_t)\dot{c}_t}{u'(c_t)} = -(r-\rho)$$

$$\dot{a}_t = ra_t + w - c_t$$

- Alternatively, we can solve for $\dot{c}_t$ (using $\ddot{a}_t$) and find a second-order differential equation.
- To characterize a solution, we need an initial and a terminal condition.
  - Initial condition: a_0 given.
  - Terminal condition: $V_T(a_T)$ or transversality in the case of infinite time.
- Note that the intertemporal elasticity of substitution is equal to

$$\frac{1}{\sigma} = -\frac{u'(c_t)}{u''(c_t)c_t}$$

where σ is the coefficient of risk aversion. Then, the Euler equation: $\dot{c}_t/c_t = (r-\rho)/\sigma$.

Digression: Solution of Differential Equations

- Suppose a first-order linear non-homogeneous differential equation:

$$\dot{y}(t) + g(t)y(t) = f(t),$$

  - where $g(t)$ and $f(t)$ are parameters that may or may not depend on t.
- The solution is given by the following formula:

$$y(t) = \frac{\int_{t_0}^{t} u(s)f(s)ds + C}{u(t)}$$

  - where $u(t)$ is the integration factor given by:

$$u(t) = \exp\left(\int_{t_0}^{t} g(s)ds\right),$$

  - t_0 the initial time and C an arbitrary constant (which can be found with an initial condition $y(0)$, or some other condition).

Example: Consumption and Saving

- Using the formulas, we can solve the EE (an ODE) $\dot{c}_t = c_t(r-\rho)/\sigma$:

$$g(t) \equiv \frac{\rho - r}{\sigma}, \qquad f(t) \equiv 0 \qquad \text{and} \qquad t_0 = 0.$$

- The integration factor is $u(t) = \exp\left(\int_0^t (\rho - r)/\sigma ds\right) = e^{\frac{(\rho-r)t}{\sigma}}$.
- Therefore, for an arbitrary constant κ_1, we have the solution:

$$c_t = e^{\frac{(r-\rho)t}{\sigma}} \kappa_1.$$

- When $t = 0$, we can find that the constant is equal to consumption in period 0: $\kappa_1 = c_0$.

Example: Consumption and Saving

- Substituting c_t in the budget constraint:

$$\dot{a}_t = ra_t + w - c_t \qquad \Rightarrow \qquad \dot{a}_t - ra_t = w - e^{\frac{(r-\rho)t}{\sigma}} c_0$$

- We can apply the formulas:

$$g(t) \equiv -r, \qquad f(t) \equiv w - e^{\frac{(\rho-r)t}{\sigma}} c_0 \qquad \text{and} \qquad u(t) = \exp\left(\int_0^t -r ds\right) = e^{-rt}.$$

- The optimal saving a_t for an arbitrary constant κ_2 is:

$$a_t = e^{rt}\left[\int_0^t e^{rs}\left(w - e^{\frac{(r-\rho)s}{\sigma}} c_0\right) ds + \kappa_2\right]$$

Example: Consumption and Saving

- Solving the integral (assume $\sigma = 1$ for simplicity):

$$a_t = e^{rt}\left[\int_0^t (we^{-rs} - c_0 e^{-\rho s})ds + \kappa_2\right] = e^{rt}\left[\frac{w}{r}(1 - e^{-rt}) - \frac{c_0}{\rho}(1 - e^{-\rho t}) + \kappa_2\right]$$

- Substituting $t = 0$, we find that the constant is equal to the initial condition: $\kappa_2 = a_0$.
- We still need c_0. Since we are in finite time, we use the final condition: $a_T = 0$:

$$0 = e^{rT}\left[\frac{w}{r}(1 - e^{-rT}) - \frac{c_0}{\rho}(1 - e^{-\rho T}) + a_0\right]$$

$$c_0 = \frac{\rho}{1 - e^{-\rho T}} \underbrace{\left[\frac{w}{r}(1 - e^{-rT}) + a_0\right]}_{\text{present value of permanent income}}$$

- In infinite time, we could use the *no-Ponzi* condition as a terminal condition.

Neoclassical Growth in Continuous Time: The Ramsey-Cass-Koopmans Model

Environment, Preferences and Technology

- Infinite horizon and continuous time. Representative household with instantaneous utility $u(c_t)$.
  - $u(c_t)$ strictly increasing, concave, twice differentiable, satisfies Inada conditions.
- **Demographics**: $L_0 = 1$ and population growth at rate n: $L_t = e^{nt}$.
- All households supply labor inelastically (i.e., no work-leisure decision).
- The firm's problem is static so it is the same as in discrete time:

$$r_t = F_K(K_t, L_t) = f'(k_t)$$
$$w_t = F_L(K_t, L_t) = f(k_t) - k_t f'(k_t)$$

Environment and Preferences

- Recall per-capita consumption: $c_t = \frac{C_t}{L_t}$, where C_t is aggregate consumption.
- Utility function:

$$\int_0^\infty e^{-\rho t} L_t u(c_t) dt = \int_0^\infty e^{-(\rho - n)t} u(c_t) dt$$

- Assumption to ensure bounded integral: $\rho > n$.

Household Problem

- The (aggregate) budget constraint is:

$$\dot{\mathcal{A}}_t = r_t \mathcal{A}_t + w_t L_t - C_t,$$

  where $\mathcal{A}_t$ is aggregate asset quantity. Define $a_t = \mathcal{A}_t / L_t$ and we have the per capita budget constraint:

$$\dot{a}_t = (r_t - \delta - n) a_t + w_t - c_t.$$

- As we have seen, equilibrium in the asset market $a_t = k_t$ (but not necessarily in models with government bonds or other risky assets).
- And the *no-Ponzi game* condition in continuous time:

$$\lim_{t \to \infty} a_t \exp\left(- \int_0^t (r_s - \delta - n) ds \right) \geq 0.$$

Household Problem

- The household's problem:

$$
\begin{aligned}
& \max_{c_t \geq 0} \int_0^\infty e^{-(\rho-n)t} u(c_t) dt \\
s.t. \quad & \dot{a}_t = (r_t - \delta - n) a_t + w_t - c_t, \\
& a_0 \text{ given}, \\
& \lim_{t \to \infty} a_t \exp\left(- \int_0^t (r_s - \delta - n) ds \right) \geq 0.
\end{aligned}
$$

Equilibrium

Definition: Competitive equilibrium (sequential) consists of allocations for the households $\{c_t, a_t\}_{t=0}^{\infty}$, allocations for the firms $\{K_t, L_t\}_{t=0}^{\infty}$, and prices $\{r_t, w_t\}_{t=0}^{\infty}$ where:

1. Given prices and $a_0 = K_0/L_0$, allocations $\{c_t, a_t\}_{t=0}^{\infty}$ solve the household's problem.
2. Given prices, allocations $\{K_t, L_t\}_{t=0}^{\infty}$ solve the firm's problem:

$$\max_{K_t, L_t} F(K_t, L_t) - r_t K_t - w_t L_t$$

3. Market clearing for the labor, capital, and goods markets.

$$\begin{aligned} e^{nt} L_0 &= L_t \\ a_t L_t &= K_t \\ F(K_t, L_t) &= \dot{K}_t + \delta K_t + L_t c_t \end{aligned}$$

Characterizing the Equilibrium

- The (current value) Hamiltonian of the HH's problem:

$$\hat{H}(a_t, c_t, \mu_t) = u(c_t) + \mu_t(a_t(r_t - \delta - n) + w_t - c_t)$$

- Necessary conditions (together with the no-Ponzi and state LOM):

$$\begin{aligned} u'(c_t) &= \mu_t \\ \mu_t(r_t - \delta - n) &= -\dot{\mu}_t + (\rho - n)\mu_t \end{aligned}$$

- Implying the Euler equation:

$$\frac{u''(c_t)\dot{c}_t}{u'(c_t)} = -(r_t - \delta - \rho).$$

or using the intertemporal substitution elasticity: $1/\sigma(c_t) = -u'(c_t)/(u''(c_t)c_t)$:

$$\frac{\dot{c}_t}{c_t} = \frac{(r_t - \delta - \rho)}{\sigma(c_t)}.$$

Characterizing the Equilibrium

- Substituting r_t:

$$\frac{\dot{c}_t}{c_t} = \frac{(f'(k_t) - \delta - \rho)}{\sigma(c_t)}.$$

- And the *market clearing* (derived from the budget constraint):

$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t$$

- The solution $\{c_t, k_t\}_{t=0}^{\infty}$ is characterized by the system of differential equations, together with the initial/terminal conditions k_0 and TVC ($\lim_{T\to\infty} e^{-\rho T}\mu_T k_T = 0$).

- Note that welfare theorems are satisfied and the Planner's solution equals the decentralized equilibrium.

Steady State

- Steady state: variables are constant over time, $\dot{k}_t = 0$ and $\dot{c}_t = 0$.
- Via SS we can find k_{ss} as a function of f, ρ, and δ (does not depend on the form of the utility function!):

$$\underbrace{f'(k_{ss})}_{r_{ss}} - \delta = \rho > n$$

- Define the *Golden Rule* as the capital that maximizes consumption:

$$\frac{dc}{dk} = f'(k_{ss}) - (\delta + n) = 0$$

- That is, the capital chosen by the Planner **is lower** than the *Golden Rule*. This happens because the Planner considers that households discount future consumption.

Steady State

- Unlike the Solow model, in RCK k_{ss} does not depend on population growth!
  - Ramsey: $f'(k_{ss}) = \delta + \rho$,
  - Solow (cns time): $\dfrac{f(k_{ss})}{k_{ss}} = \dfrac{\delta + n}{s}$,

  Note the connection between the savings rate s in Solow and the discount rate in RCK $\Rightarrow \uparrow \rho$ more impatient and lower capital accumulation.
- Once we have k_{ss} computing the rest is easy:
  - Aggregate resource constraint: $c_{ss} = f(k_{ss}) - (\delta + n)k_{ss}$.
  - Savings rate:

$$c_{ss} = (1 - s_{ss})f(k_{ss}) \Leftrightarrow s_{ss} = \frac{(\delta + n)k_{ss}}{f(k_{ss})}$$

Steady State

- **Example**: Use $f(k_t) = Ak_t^\alpha$ and do comparative statics of the effects of A, δ, n, and ρ on c_{ss} and k_{ss}.

$$k_{ss} = \left(\frac{\alpha A}{\delta + \rho}\right)^{1/(1-\alpha)}$$

$$c_{ss} = k_{ss}(Ak_{ss}^{\alpha-1} - (\delta + n)) = k_{ss}\left(\frac{(\delta + \rho) - \alpha(\delta + n)}{\alpha}\right)$$

- Since $\rho > n$ and $\alpha < 1$, consumption in the SS is a fraction of capital in the SS.
  - $\partial k_{ss}/\partial A > 0$ and $\partial c_{ss}/\partial A > 0$
  - $\partial k_{ss}/\partial \rho < 0$ and $\partial c_{ss}/\partial \rho < 0$
  - $\partial k_{ss}/\partial \delta < 0$ and $\partial c_{ss}/\partial \delta < 0$
  - $\partial k_{ss}/\partial n = 0$ and $\partial c_{ss}/\partial n < 0$

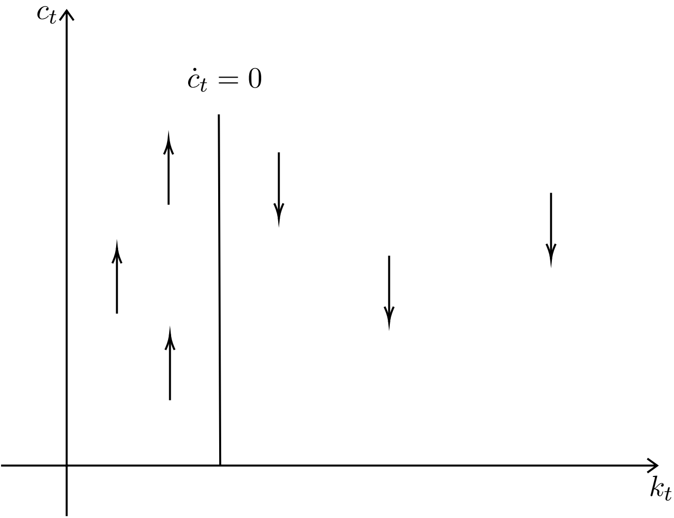
Transition Dynamics

- Remember that the equilibrium is characterized by the equations (+ TVC and k_0):

$$\frac{\dot{c}_t}{c_t} = \frac{(f'(k_t) - \delta - \rho)}{\sigma(c_t)},$$
$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t.$$

- How can we analyze the dynamics of the system outside the steady state? $\Rightarrow$ Phase diagram.
- We will also show that the system is *saddle-path stable*: there exists a unique trajectory $\{k_t, c_t\}$ that converges to the steady state.
  - Given the state k_0, the control c_0 (or alternatively μ_0) adjusts instantly to the unique trajectory. That's why the control variable is known as the ***jump variable***.
  - For example, if an unexpected policy change occurs, the consumer adjusts c_t to the optimal path immediately while the state k necessarily follows the LOM.

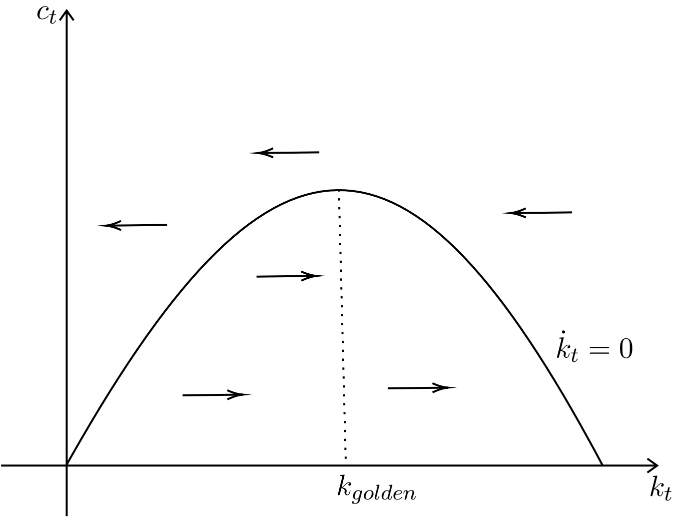
Phase Diagram



$$\frac{\dot{c}_t}{c_t} = \frac{(f'(k_t) - \delta - \rho)}{\sigma(c_t)}$$

- If $\uparrow k_t \Rightarrow \downarrow f'(k_t) \Rightarrow \downarrow \dot{c}$.
- The vertical line represents the space where consumption is constant.
- The line is vertical as depends only on capital (and not on consumption): $f'(k_t) = \delta + \rho$.

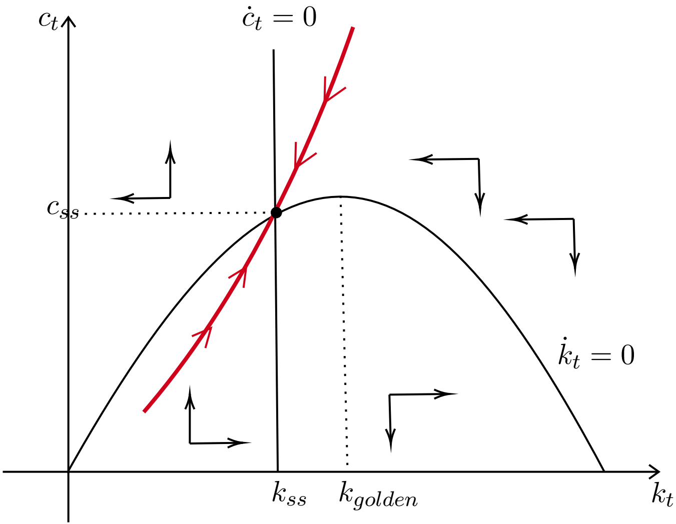
Phase Diagram



$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t$$

- If $\uparrow c_t \Rightarrow \downarrow \dot{k}$.
- Capital is constant at the inverted U-shape line: $f(k_t) - (\delta + n)k_t$.
- k_{golden} is the point that maximizes consumption.

Phase Diagram



- Given k_0, c_0 "jumps" to the *stable-path*.
- If $c_0' > c_0$ the path converges to $k = 0$ and $c > 0$: violates the feasibility condition.
- If $c_0'' < c_0$ the path converges to $c = 0$ and $k > 0$: violates the TVC.
- There is only one path that converges to the steady state.

Local Stability

- Another way to check the *Saddle-path Stability* of the system is to look at the local stability conditions.
- Linearize the system equations using Taylor expansion around the Steady State:

$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t \Rightarrow \dot{k} = (f'(k_{ss}) - (\delta + n))(k - k_{ss}) - (c - c_{ss})$$
$$\dot{c}_t = c_t \frac{(f'(k_t) - \delta - \rho)}{\sigma} \Rightarrow \dot{c} = c_{ss} \frac{f''(k_{ss})}{\sigma}(k - k_{ss}) + \underbrace{\frac{(f'(k_{ss}) - \delta - \rho)}{\sigma}}_{=0}(c - c_{ss})$$

- Write the system in the form:

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} f'(k_{ss}) - \delta - n & -1 \\ c_{ss} f''(k_{ss})/\sigma & 0 \end{bmatrix} \begin{bmatrix} k - k_{ss} \\ c - c_{ss} \end{bmatrix}$$

Local Stability

Theorem (Acemoglu 7.19)

Consider the system $\dot{x}_t = G(x_t)$ where G is continuously differentiable and x_0 given. The steady state is $G(x^*) = 0$ and define $A = DG(x^*)$, D is the jacobian of G.
Suppose m eigenvalues of A have negative real parts while $n - m$ have positive real parts.
Then there exists a m-dimensional manifold (i.e., a topological space) in the neighborhood of the steady state, such that from any x_0 on that manifold, there is a unique $x_t \to x^*$.

- In our case if the matrix

$$A = \begin{bmatrix} f'(k_{ss}) - \delta - n & -1 \\ c^* f''(k_{ss})/\sigma & 0 \end{bmatrix}$$

has $m = 1$ negative eigenvalues, then there exists a line (manifold dimension $m = 1$) of points (c, k) that converges to the steady state.

Local Stability

- We want to find the eigenvalues λ such that $Ax = \lambda x$. In this case we have $\det(A - \lambda I)x = 0$

$$\det \begin{bmatrix} f'(k_{ss}) - \delta - n - \lambda & -1 \\ c^* f''(k_{ss})/\sigma & 0 - \lambda \end{bmatrix} = 0$$

$$\det(A - \lambda I) = -\lambda[f'(k_{ss}) - \delta - n - \lambda] + c_{ss} f''(k_{ss})/\sigma = 0$$

so

$$\lambda = [f'(k_{ss}) - \delta - n \pm \sqrt{(f'(k_s) - \delta - n)^2 - 4c_{ss} \underbrace{f''(k_{ss})}_{<0} /\sigma}]/2$$

- Since $\sqrt{(f'(k_{ss}) - \delta - n)^2 - 4c_{ss} f''(k_{ss})/\sigma} > f'(k_{ss}) - \delta - n$, there exists exactly one negative eigenvalue.

Balanced Growth Path

- For the model to be consistent with the Facts of Kaldor it has to have long-run growth.
- **Balanced Growth Path**: All variables grow at a constant rate.
- Suppose a technology with *Labor-augmenting Technological Change*:

$$Y_t = F(K_t, A_t L_t), \quad \text{where } A_t = A_0 e^{gt}$$

  and g is the growth rate.
- We redefine the variables in labor efficient units so that they remain stationary: $\tilde{y}_t = y_t/A_t$, $\tilde{k}_t = k_t/A_t$, $\tilde{c}_t = c_t/A_t$.

Balanced Growth Path

- Note that the growth of the new variables is the per capita growth rate minus the technological advance:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{c}}{c} - \frac{\dot{A}_t}{A_t} = \frac{\dot{c}}{c} - g \quad \text{and} \quad \frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{k}}{k} - g$$

- Using this argument, the fact that F is CRS $(F(k, A) = AF(\tilde{k}, 1))$, and the definition of $\dot{k}$:

$$\begin{aligned} \frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{F(\tilde{k}_t, 1)A_t - (\delta + n)\tilde{k}_t A_t - \tilde{c}_t A_t}{k_t} - g \\ \dot{\tilde{k}} &= f(\tilde{k}_t) - (\delta + n + g)\tilde{k}_t - \tilde{c}_t \end{aligned}$$

Balanced Growth Path

- And the stationary Euler Equation:

$$
\begin{aligned}
\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{(F_k(k_t, A_t) - \delta - \rho)}{\sigma(c_t)} - g \\
&= \frac{(f'(\tilde{k}_t) - \delta - \rho)}{\sigma(c_t)} - g,
\end{aligned}
$$

where we used the fact that

$$
\frac{\partial F(k, A)}{\partial k} = \frac{\partial F(A\tilde{k}, A)}{\partial \tilde{k}} \frac{\partial \tilde{k}}{\partial k} = A f'(\tilde{k}) \frac{1}{A} = f'(\tilde{k}).
$$

- By a similar argument we have that $r_{ss} = f'(\tilde{k})$ is constant in the long run (Kaldor's Facts), i.e., $r_t \to r_{ss}$.

Balanced Growth Path

- The only way for $\dot{\tilde{c}} = 0$, is if per capita consumption grows at a constant rate in the long run: $\dot{c}_t / c_t \to g$.
- By the Euler Equation, this implies that $\sigma(c_t) \to \sigma$.
- **Condition for BGP** is that the elasticity of marginal utility of consumption is asymptotically constant. Or alternatively, that the intertemporal elasticity of substitution is asymptotically constant.
- That's why the CRRA utility is so commonly used:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

Balanced Growth Path

- Given that $\sigma(c_t) \to \sigma$ is constant in the long run. The condition for utility to be bounded now is: $\rho - n > g(1-\sigma)$. Intuition:

$$\int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\sigma}}{1-\sigma} dt$$

$$\int_0^\infty e^{-(\rho-n)t} \frac{(\tilde{c}_t A_0 e^{gt})^{1-\sigma}}{1-\sigma} dt$$

$$\int_0^\infty e^{-(\rho-n-g(1-\sigma))t} \frac{(\tilde{c}_t A_0)^{1-\sigma}}{1-\sigma} dt$$

Balanced Growth Path

- The solution now is the system of equations:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{(f'(\tilde{k}_t) - \delta - \rho - g\sigma)}{\sigma}$$
$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - (\delta + n + g)\tilde{k}_t - \tilde{c}_t$$

- Along with the initial condition $\tilde{k}_0$ and the TVC:

$$\lim_{t\to\infty} e^{-(\rho-n-g(1-\sigma))t} u'(\tilde{c}_t)\tilde{k}_t = 0$$

Steady State

- Note that now the steady-state capital depends on the form of utility (σ):

$$f'(\tilde{k}_{ss}) = \delta + \rho + g\sigma,$$

  which implies that: $r_{ss} = \delta + \rho + g\sigma$!

- $\uparrow \sigma \rightarrow$ lower intertemporal elasticity of substitution $\rightarrow \downarrow \tilde{k}_{ss}$.
- In a way, very similar to Solow:
  - $\tilde{k}$ is endogenous and depends on δ, g, and discount/IES determines savings (Solow: exogenous savings rate, but depends on n).
  - Long-run per capita growth is exogenous and given by g (just like in Solow).

Example

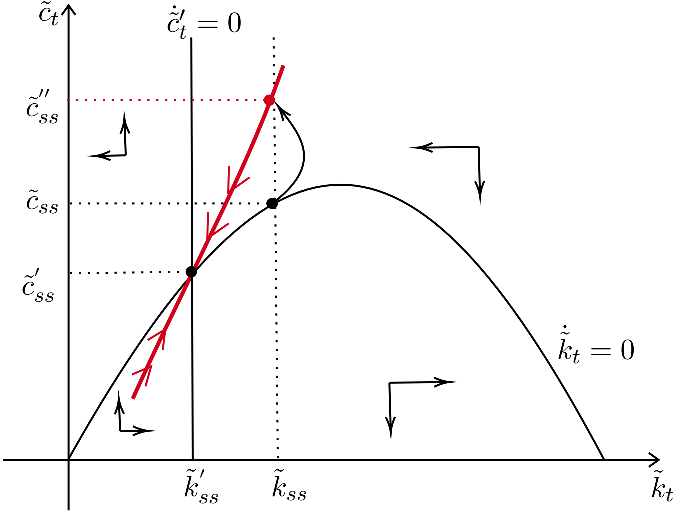
- Consider CRRA utility and Cobb-Douglas production function, $Y_t = F(K_t, A_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}$:
  - $\tilde{y}_t = \tilde{k}^{\alpha}$, where for an arbitrary aggregate variable X, $\tilde{x} = X/(AL)$.
  - $r = f'(\tilde{k}) = \alpha \tilde{k}^{\alpha - 1}$.
- The Euler Equation and the resource constraint:

$$\begin{aligned} \frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{(r_t - \delta - \rho)}{\sigma(c_t)} - g = \frac{1}{\sigma}(\alpha \tilde{k}^{\alpha-1} - \delta - \rho - \sigma g) \\ \dot{\tilde{k}} &= f(\tilde{k}_t) - (\delta + n + g)\tilde{k}_t - \tilde{c}_t = \tilde{k}^{\alpha} - (\delta + n + g)\tilde{k}_t - \tilde{c}_t \end{aligned}$$

- Steady state:

$$\tilde{k}_{ss} = \left(\frac{\alpha}{\delta + \rho + \sigma g}\right)^{1/(1-\alpha)} \quad \text{and} \quad \tilde{c}_{ss} = \left(\frac{\alpha}{\delta + \rho + \sigma g}\right)^{\frac{1}{1-\alpha}} \left(\frac{(\delta + \rho + \sigma g) - \alpha(\delta + n + g}{\alpha}\right.$$

Comparative Dynamics: Increase in ρ

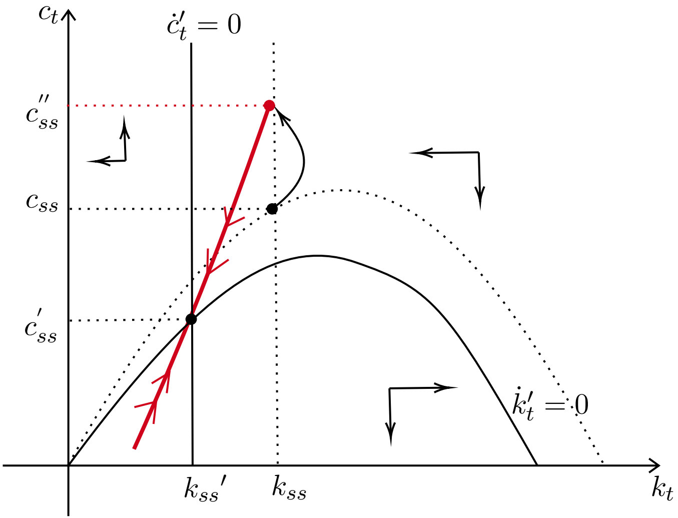


- Suppose the economy is in the SS and ρ increases.

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{(f'(\tilde{k}_t) - \delta - \rho - g\sigma)}{\sigma}$$

- The line $\dot{\tilde{c}}$ shifts to the left and $\tilde{c}$ jumps to the new *stable-path*.
- Eventually the system converges to the new SS.

Comparative Dynamics: Increase in δ



- Suppose the economy is in the SS and δ increases.

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{(f'(\tilde{k}_t) - \delta - \rho - g\sigma)}{\sigma}$$
$$\dot{\tilde{k}} = f(\tilde{k}_t) - (\delta + n + g)\tilde{k}_t - \tilde{c}_t$$

- The line $\dot{\tilde{c}}$ shifts to the left and the line $\dot{\tilde{k}}$ shifts downward.
- $\tilde{c}$ jumps, and eventually the system converges to the new SS.

Policy: Capital Tax

- Consider a small extension: the net return on capital is taxed at τ:

$$\hat{r}_t = (1-\tau)(r_t - \delta) = (1-\tau)(f'(\tilde{k}_t) - \delta)$$

- The aggregate tax revenue is distributed via a lump-sum transfer $\tilde{t}$, so the budget constraint becomes:

$$\dot{\tilde{a}}_t = \tilde{a}_t(\hat{r}_t - n - g) - \tilde{c}_t + w_t + \tilde{t}_t,$$

  where the adjusted transfer equals the revenue: $\tilde{t} = \tau(r_t - \delta)\tilde{a}_t$.

- The tax distorts capital accumulation, and therefore the Euler Equation becomes:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{((1-\tau)(f'(\tilde{k}_t) - \delta) - \rho - g\sigma)}{\sigma}$$

Policy

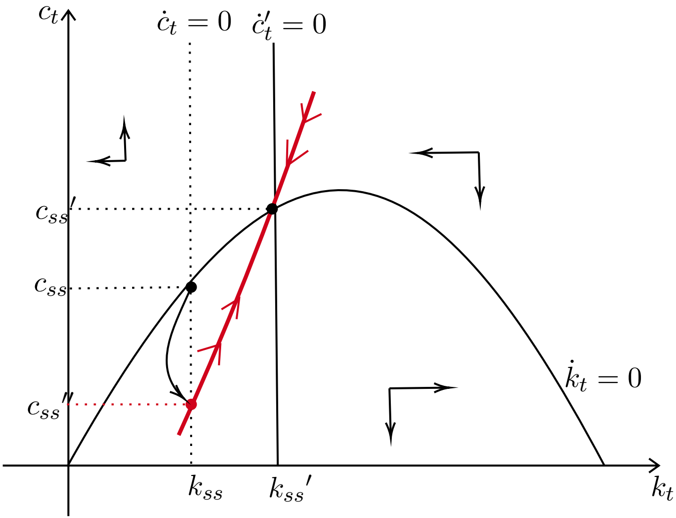
- But since the tax is refunded back to the consumer (there is no government!), the economy's resource constraint does not change (verify this!):

$$\dot{\tilde{k}} \ = f(\tilde{k}_t) - (\delta + n + g)\tilde{k}_t - \tilde{c}_t$$

- Given the disincentive to capital accumulation, the capital in the steady state will be lower:

$$f'(\tilde{k}_{ss}) = \delta + \frac{\rho + \sigma g}{1 - \tau}$$

Policy: Decrease in τ



- Decrease in τ: increase incentives to accumulate capital
- The increase in the savings rate reduces consumption initially.
- But as accumulates capital, it increases capital/production/consumption.

Solving for the Equilibrium Path Numerically

- We saw that the Neoclassical Growth Model cannot be solved analytically (except in very special cases). This means, we must solve it numerically.

- The simplest method to solve the transition dynamics is using **shooting algorithm**.

- The main idea:
  - We have the initial condition k_0 (or the initial steady state) and the final steady state (k_{ss} and c_{ss}).
  - We can use the solution system to connect the initial to the terminal value. In discrete time, the system is:

$$u'(c_t) = (f'(k_{t+1}) + 1 - \delta)\beta u'(c_{t+1}) \qquad \text{(Euler Equation)}$$

$$c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t \qquad \text{(Resource Constraint)}$$

Shooting Algorithm

- **Shooting algorithm:** We guess an initial value of c_0 and use the system of equations to simulate the sequence $\{k_{t+1}, c_t\}_{t=0}^{S}$ until a period S.
  - If we stop at the final steady state, c_0 is the solution;
  - otherwise, we try another c_0.
- It is an intuitive algorithm that works to solve systems of difference (or differential) equations with two conditions. It works in discrete and continuous time.
- Recall the phase diagram. We are trying to guess the initial jump of the control variable.
- If we miss the initial jump of c_0, the path diverges from the steady state.
- We will use it more later when we study fiscal policy.

Shooting Algorithm

(i) Solve for the final steady state: k_{ss} and c_{ss}.

(ii) Select a sufficiently long time period S (so that the economy reaches the SS), and guess an initial consumption solution candidate c_0.

(iii) Use the *Resource Constraint* and k_0 to compute k_1. Use c_0 and k_1 in the Euler Equation to compute c_1. Continue using c_t and k_t to find c_{t+1} and k_{t+1}.

(iv) Proceed until period S to find the candidate solution sequence: $\{\hat{k}_{t+1}, \hat{c}_t\}_{t=0}^{S}$.

(v) Compute $\hat{k}_S - k_{ss}$. If $\hat{k}_S > k_{ss}$, increase the guess c_0 and try again; If $\hat{k}_S < k_{ss}$, decrease the guess c_0 and try again.

(vi) Proceed until finding a c_0 such that $\hat{k}_S \approx k_{ss}$.

Taking Stock

- Neoclassical growth model: explains the convergence process among different countries.
  - Many conclusions are similar to the Solow model.
- Endogenous savings bring new insights into the impact of preferences, taxation, etc., on long-term growth.
- For the Balanced-Growth Path, we need preferences with constant elasticity of substitution.
- Does not explain very long-term growth $\Rightarrow$ grows at the exogenous rate g.
  - Motivation to develop endogenous growth models.