# Macroeconomics I
# Problem Set 4

1. **(OLG with Distortionary Taxes).** Consider the standard OLG model where the agent works when young and consumes savings when old. All assumptions are usual and follow the model seen in class. The utility is:

$$\ln(c_t^1) + \beta \ln(c_{t+1}^2).$$

The agent's income is subject to two types of taxes: on labor $\tau_w$ and on capital $\tau_k$. Suppose the tax revenue is thrown into the ocean. The agent's budget constraints when young and old are:

$$\begin{aligned} c_t^1 + s_t &\leq w_t(1-\tau_w) \\ c_{t+1}^2 &\leq [1 + (r_{t+1} - \delta)(1-\tau_k)]s_t. \end{aligned}$$

The production function is: $Y_t = K_t^\alpha L_t^{1-\alpha}$, with $\alpha \in (0,1)$ and the population grows $L_t = (1+n)^t L_0$ where $L_0 = 1$. The initial old generation has $k_0 > 0$ given.

   (a) Solve the household's problem. Find $c_t^1$, $c_{t+1}^2$, and $s_t$. Why does savings not depend on $\tau_k$?

   (b) Use the solution to the firm's problem (standard) and the equilibrium condition in the asset market to find a first-order difference equation characterizing the optimal sequence of capital.

   (c) Write down the equation determining capital in the steady state. How does it depend on $\tau_w$? Explain intuitively why the labor tax has a different effect compared to the standard neoclassical growth model (without leisure in the utility function and in infinite horizon).

2. **(Intergenerational Altruism).** Consider an OLG model where a mass of individuals with unit measure live two periods: childhood and adulthood. Individuals have warm-glow preferences, i.e., they value the bequests left for their children in their utility. In adulthood, the individual receives the inheritance from their parents (and rents it out as capital to firms), has children, works, chooses the inheritance for their children, and dies. The utility in childhood is not relevant (imagine that children's consumption is already incorporated into parents' consumption). The utility of an individual $i$ reaching adulthood at $t$ is:

$$\ln(c_t^i) + \beta \ln(b_t^i),$$

where $b_t^i$ is the bequest left for their children and $\beta \in (0,1)$. The budget constraint is

$$c_t^i + b_t^i = w_t + (1 + r_t - \delta) b_{t-1}^i,$$

and $b_0^i = b_0 > 0$ given.[1] The production side is standard: $Y_t = K_t^\alpha L_t^{1-\alpha}$ with $\alpha \in (0,1)$. There is no population growth, and $L_0 = 1$.

(a) Characterize the equilibrium of the economy. That is, solve the household's problem, the firm's problem, and write down the set of equations characterizing the optimal allocations and prices. How does capital evolve in this economy?

(b) Find the steady state of $k_{ss}, c_{ss}, b_{ss}$, and $y_{ss}$ as functions of the parameters.

(c) Now suppose the individual values the TOTAL utility of their children (not just the bequest left). The utility of an individual reaching adulthood at $t$ is:

$$U_t = \ln(c_t) + \beta U_{t+1}.$$

Everything else remains the same (add a no-Ponzi and a TVC). Characterize the equilibrium of the economy.

(d) Suppose full depreciation, $\delta = 1$ (to simplify). Compare the steady-state capital in the two cases. Under what conditions are they equal? Explain intuitively how your answer would change if we used a CRRA function.

3. **(Families, Endogenous Fertility, and Human Capital).**[2]

(a) *Barro-Becker Endogenous Fertility Model* with human capital. A (uniparental) household derives utility from consumption ($c$), their number of children ($n$), and the future income of their children ($y'$):

$$\ln(c) + \gamma_n \ln(n) + \gamma \ln(y'). \tag{1}$$

Denote the household's human capital as $H$ and the time spent on production as $\ell$. The production of consumption is $c = y\ell$, where $y = AH$ is the total income of the household if they worked full-time.

The household's time is allocated to production and raising children. Denote $e$ as the education given to the children and $\phi$ a fixed cost (in time) in raising children. The family's time constraint is:

$$\ell + n(\phi + e) \leq 1. \tag{2}$$

The human capital of children evolves according to the parents' human capital and the invested education: $H' = (Be)^\theta H$. Suppose that: $\gamma_n > \gamma\theta$.

i. Show that the equilibrium education and fertility are:

$$e^* = \frac{\phi\gamma\theta}{\gamma_n - \gamma\theta} \qquad \text{and} \qquad n^* = \frac{\gamma_n - \gamma\theta}{\phi(1+\gamma_n)}. \tag{3}$$

[1] Note that $b_0$ is the same for all individuals, and you can ignore the index $i$. See Acemoglu's book for the case where individuals start with heterogeneous inheritance.

[2] Question based on the chapter from Doepke and Tertilt (2016, Handbook of Macroeconomics): *Families in Macroeconomics.*

ii. What is the (per-capita) growth rate of the economy (in terms of parameters)? What is the model's prediction for growth and fertility when there is a gradual increase in the return on human capital ($\theta$)? What is the impact of a fertility restriction policy (like China's one-child policy) on per-capita economic growth?

(b) Bargaining power in *2-parent* households. Now consider a household consisting of a husband, a wife, a son, and a daughter (we will ignore fertility decisions). Households share consumption, but men and women disagree on how much they care about the welfare of their children. The household's utility is the sum of the parents' utility:

$$\lambda_f[\ln(c) + \gamma_f \ln(y')] + (1 - \lambda_f)[\ln(c) + \gamma_m \ln(y')], \tag{4}$$

where $\lambda_f$ is the woman's bargaining power in family decisions, $\gamma_f$ and $\gamma_m$ are the parameters of altruism of the woman and the man.

To simplify, consider that only women raise children. Women's time constraint is:

$$\ell + e_f + e_m \leq 1,$$

where $e_f$ and $e_m$ are the education invested in daughters and sons. Women and men are imperfect substitutes in the production of goods and human capital:

$$c = A(\ell H_f)^\alpha H_m^{(1-\alpha)}, \qquad H'_f = (Be_f)^\theta H_f^\beta H_m^{1-\beta}, \quad \text{and} \quad H'_m = (Be_m)^\theta H_f^\beta H_m^{1-\beta},$$

and the total income if the woman works full-time is: $y = A(H_f)^\alpha H_m^{(1-\alpha)}$.

i. Show that the equilibrium investment in education is:

$$e_f^* = \frac{\theta\alpha\delta}{\alpha + \delta\theta} \qquad \text{and} \qquad e_m^* = \frac{\theta(1-\alpha)\delta}{\alpha + \delta\theta},$$

where $\delta \equiv \lambda_f\gamma_f + (1 - \lambda_f)\gamma_m$.

How does an increase in the relative productivity of women ($\alpha$) change the gender education gap (i.e., $e_f/e_m$) and the time spent on children?

ii. Derive the per-capita growth rate of this economy ($y'/y$). Suppose that women are more altruistic towards children: $\gamma_f > \gamma_m$. How does an increase in women's bargaining power $\lambda_f$ alter the growth rate of this economy?