Macroeconomics I
Problem Set 4

1. **(Recursive Equilibrium with Taxation).** Consider an infinite horizon economy with a continuum of individuals indexed by $i \in [0, 1]$. The budget constraint of individual i is:

$$c_t^{(i)} + k_{t+1}^{(i)} = (1 - \tau)Ak_t^{(i)},$$

meaning, each individual has a linear technology in their own capital, $k_t^{(i)}$. Production is taxed at a rate τ. Individuals can use their net revenue from production for consumption, $c_t^{(i)}$, or for investment, $k_{t+1}^{(i)}$. Capital fully depreciates after use. Government spending $G_t > 0$ follows a first-order Markov process. The government balances its budget in each period, meaning it chooses τ_t at each t to satisfy:

$$\tau_t AK_t = G_t,$$

where K_t is the aggregate capital.[1] Individuals have rational expectations about future taxation, and their preferences are defined as:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t^{(i)}),$$

where u satisfies the usual assumptions.

   (a) Social Planner:

      i. Write the resource constraint at time t of a social planner who chooses the sequence of aggregate variables $\{C_t, K_{t+1}\}_{t=0}^{\infty}$ and considers the stochastic sequence $\{G_t\}_{t=0}^{\infty}$ as given.

      ii. Write the Bellman equation for this social planner (do not forget G).

      iii. From the Bellman equation, find the Euler equation for the social planner's problem (assume the usual regularity conditions are satisfied).

   (b) Recursive Competitive Equilibrium (RCE):

      i. Write the Bellman equation for an individual i.

      ii. Find the individual's Euler equation in the RCE.

   (c) Compare the Euler equations of the social planner's problem and the individual's problem and discuss whether capital accumulation is efficient in the RCE, or if it is below or above the efficient level.

[1] Suppose K_0 is sufficiently large so that, in equilibrium, it is always possible for the government to pay its bills.

2. **(Growth under Stochastic Depreciation)**. Consider an economy where the final good is produced according to the production function $y_t = F(k_t)$, where k_t is the capital. The representative agent chooses the consumption sequence according to the preferences:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right],$$

where $\beta \in (0, 1)$ and $u(.)$ has the usual properties. The final good can be used for consumption, $c_t \geq 0$, or investment, $i_t \geq 0$. The capital's law of motion is stochastic and follows:

$$k_{t+1} = k_t(1 - z_{t+1}) + i_t,$$

where $z_{t+1} \in (0, 1)$ is a random variable that follows a first-order Markov process.

(a) Describe the benevolent social planner's problem in Dynamic Programming form: What are the state variables, control variables, state transition law, feasible set, and return function. Write the Bellman equation.

(b) Suppose z_t follows an *iid* process: find the state variables and write the Bellman equation.

(c) Suppose z_t follows a second-order Markov chain: find the state variables and write the Bellman equation.

3. **(Markov-perfect Equilibrium)**. Consider the *cake-eating problem* of an agent with cake x_0 (imagine the cake is a depletable resource like the Amazon forest). Every period, the agent consumes a part of the cake: $x_{t+1} = R(x_t - c_t)$, where $c_t > 0$ is the consumption and $R > 0$ is the gross cake growth rate (or reforestation rate). The agent chooses the consumption sequence that maximizes total utility:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t),$$

where $\beta \in (0, 1)$.

(a) Write the Bellman equation for the agent. Suppose the Bellman equation has the form $V(x) = C + D \ln x$. Use the method of undetermined coefficients and find the coefficients C, D, and the decision rule $c^*(x)$.

(b) *Markov-perfect Equilibrium.* Now suppose there are *two* agents eating the cake. Each period, they decide how much cake to eat. The law of motion now is:

$$x_{t+1} = R(x_t - c_t^1 - c_t^2),$$

where c^i is agent i's consumption, $i \in (1, 2)$. Both agents have logarithmic utility and discount the future by β. We will look for a *Markov-perfect Equilibrium* (MPE), a pair of symmetric strategies $c^1(x) = c^2(x)$ where each agent responds optimally to the other agent's decisions.[2]

[2] In the MPE, agents condition their strategy only on the payoff relevant state, in this case x.

i. Write the Bellman equation characterizing player 1's problem. Consider player 2's strategy (decision rule) $c^2(x)$ as a continuously differentiable function. From the Bellman equation, find the (generalized) Euler equation of agent 1 and interpret it briefly.

ii. Find the symmetric MPE (Hint: assume $c^1 = c^2$ and use the method of undetermined coefficients to find the decision rule $c^{1,*}(x) = c^{2,*}(x)$).

iii. How does the extraction rate (cake consumption) differ from the single-player case?

iv. Is the allocation in the two-player game efficient? Comment briefly (no need to use mathematical arguments).

4. **(Coding the Stochastic Growth Model)**. Modify the program of the deterministic growth model (with infinite time) to accommodate stochastic productivity. Suppose the production function is given by $y_t = A_t k_t^\alpha$, where the total factor productivity, A_t, follows a Markov chain with two states: $A_t \in \{1-a, 1+a\}$, and $Prob(A_{t+1} = A_t) = p$. Assume initially that $a = 0.05$ and $p = 0.8$.

(a) Use *Value Function Iteration* and *grid search* to find the value function numerically. Show the value functions $V(k_i, A_1)$ and $V(k_i, A_2)$ on a graph.

(b) How do the results change if: $a = 0.01$, $a = 0.1$, $p = 0.5$, and $p = 0.95$ (make one change at a time). Comment.

(c) In what sense is there a Steady State now? How could we find it? Describe briefly (Hint: remember that under certain conditions the Markov chain has a unique stationary distribution).