Macroeconomics I
Problem Set 3

1. **(Consumption and Savings with Habit Formation)**. Consider an agent whose utility depends not only on current consumption c_t, but also on past consumption c_{t-1}:[1]

$$\sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1}),$$

where u is differentiable and concave in both arguments, and is increasing in c_t. In addition to $\beta \in (0,1)$ and $c_{-1} > 0$ given. The agent is endowed with assets $a_0 > 0$ and receives an income $y > 0$ each period. Suppose the agent can save at a gross interest rate $(1 + r) = R > 1$ and that there is a borrowing limit given by $a > -\overline{A}$.

   (a) Write down this problem in Dynamic Programming form: Describe the state variables, control variable, return function, and feasible set.

   (b) Write down the Bellman equation.

   (c) Find and interpret the Euler equation.

2. **(Search for Two Husbands with Dona Flor)**. Dona Flor lives in a world that lasts two periods: $t = 0, 1$. She is looking for a husband and has two dates scheduled: one with Vadinho at $t = 0$, and another with Teodoro at $t = 1$.[2] When meeting Vadinho, she observes the *match quality*, which is a random variable x distributed uniformly on $[0, 1]$; 1 means that Vadinho is the perfect husband for her, while 0 means he is quite ordinary. If she stays with him, Dona Flor receives utility x in both periods (and cancels the date with Teodoro).

   If she decides to leave Vadinho, she receives utility σ in period 0 and meets Teodoro at $t = 1$, where $\sigma \in (0, 1)$ is given exogenously. In the encounter with Teodoro, Dona Flor observes her *match quality* with Teodoro, given by y. Suppose that y is uniformly distributed on $[0, 1]$ and is independent of x. Again, Dona Flor can decide whether to stay with Teodoro and receive utility y, or to remain single, in which case she receives $\sigma \in (0, 1)$. Dona Flor maximizes expected utility and discounts the period with $\beta \in (0, 1)$.

   (a) Write down the Bellman equations that characterize Dona Flor's problem: What are the state variables, control variable, return function, and the feasible control set.

   (b) Find Dona Flor's decision rules.

   (c) How does the decision rule at time 0 change as σ increases? Briefly explain the intuition behind the result.

[1] Preferences with habit formation are used in finance models (e.g., Campbell and Cochrane (1999)), and in DSGE models (e.g., Christiano, Eichenbaum and Evans (2005)).

[2] In Jorge Amado's original novel (1966), Vadinho is dead, and Dona Flor has to choose between Teodoro and Vadinho's spirit (!).

(d) How does the decision rule at time 0 change as β increases? Briefly explain the intuition behind the result.

(e) **Bonus (not required to submit)**: Suppose the following changes: $\sigma = 0$ and $\beta = 1$; x is distributed as before, but $y = \rho x + u$, $\rho \in (0, 1)$ and where u is a random variable uniformly distributed on $[-1, 1]$ (and independent of x).

i. Show that the value of rejecting Vadinho is

$$V_0^R(x) = \frac{\rho^2}{4}x^2 + \frac{\rho}{2}x + \frac{1}{4}.$$

ii. Explain intuitively why now V^R is increasing in x (but it wasn't before).

iii. Show that the probability that Dona Flor rejects Vadinho is increasing in ρ.

iv. For $\rho = 0.5$, find the decision rule at period $t = 0$.

3. **(Growth under Stochastic Depreciation)**. Consider an economy where the final good is produced according to the production function $y_t = F(k_t)$, where k_t is the capital. The representative agent chooses the consumption sequence according to the preferences:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right],$$

where $\beta \in (0, 1)$ and $u(.)$ has the usual properties. The final good can be used for consumption, $c_t \geq 0$, or investment, $i_t \geq 0$. The capital's law of motion is stochastic and follows:

$$k_{t+1} = k_t(1 - z_{t+1}) + i_t,$$

where $z_{t+1} \in (0, 1)$ is a random variable that follows a first-order Markov process.

(a) Describe the benevolent social planner's problem in Dynamic Programming form: What are the state variables, control variables, state law of motion, feasible set, and return function. Write the Bellman equation.

(b) Suppose z_t follows an *iid* process: find the state variables and write the Bellman equation.

(c) Suppose z_t follows a second-order Markov chain: find the state variables and write the Bellman equation.

4. **(Markov-perfect Equilibrium)**. Consider the *cake-eating problem* of an agent with cake x_0 (imagine the cake is a depletable resource like the Amazon forest). Every period, the agent consumes a part of the cake: $x_{t+1} = R(x_t - c_t)$, where $c_t > 0$ is the consumption and $R > 0$ is the gross cake growth rate (or reforestation rate). The agent chooses the consumption sequence that maximizes total utility:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t),$$

where $\beta \in (0, 1)$.

(a) Write the Bellman equation for the agent. Suppose the Bellman equation has the form $V(x) = C + D \ln x$. Use the method of undetermined coefficients and find the coefficients C, D, and the decision rule $c^*(x)$.

(b) *Markov-perfect Equilibrium.* Now suppose there are *two* agents eating the cake. Each period, they decide how much cake to eat. The law of motion now is:

$$x_{t+1} = R(x_t - c_t^1 - c_t^2),$$

where c^i is agent i's consumption, $i \in (1, 2)$. Both agents have logarithmic utility and discount the future by β. We will look for a *Markov-perfect Equilibrium* (MPE), a pair of symmetric strategies $c^1(x) = c^2(x)$ where each agent responds optimally to the other agent's decisions.[3]

i. Write the Bellman equation characterizing player 1's problem. Consider player 2's strategy (decision rule) $c^2(x)$ as a continuously differentiable function. From the Bellman equation, find the (generalized) Euler equation of agent 1 and interpret it briefly.

ii. Find the symmetric MPE (Hint: assume $c^1 = c^2$ and use the method of undetermined coefficients to find the decision rule $c^{1,*}(x) = c^{2,*}(x)$).

iii. How does the extraction rate (cake consumption) differ from the single-player case?

iv. Is the allocation in the two-player game efficient? Comment briefly (no need to use mathematical arguments).

5. **(Coding the Growth Model in Finite Time)**. Consider the "standard" neoclassical growth model in finite horizon. The production function k_t^α, the capital motion law is given by:

$$k_{t+1} = k_t(1 - \delta) + k_t^\alpha - c_t,$$

and the representative household chooses the consumption sequence to maximize the following utility function:

$$\sum_{t=0}^{T} \beta^t \log(c_t).$$

The Bellman equation of the problem is:

$$V_t(k) = \max_{k'}\{\log(k^\alpha + k(1 - \delta) - k') + \beta V_{t+1}(k')\} \quad t = 0, ..., T - 1$$

and

$$V_T(k') = \max_{k'}\{\log(k^\alpha + k(1 - \delta) - k'\},$$

with the associated policy function: $k' = g_t(k)$.

(a) Carefully describe the algorithm for finding the value function and the policy function.

[3] In the MPE, agents condition their strategy only on the payoff relevant state, in this case x.

(b) Consider $T = 50$, $\alpha = 0.3$, $\beta = 0.96$, and $\delta = 0.1$. Discretize the capital space into $n_k = 200$ equidistant points, with $k_{min} = 2k_{ss}/n_k$ and $k_{max} = 2k_{ss}$ (k_{ss} is the steady-state capital in an infinite horizon problem). Implement the algorithm in a programming language of your choice.

(c) Set $k_0 = k_{min}$. Use the policy function to simulate the optimal capital sequence $\{k^*_{t+1}\}_{t=0}^{T}$. Represent the solution in a figure. How long does it take for the optimal sequence to reach the steady state (if it does)? Change the parameters $\beta = 0.8$ and $\beta = 0.99$ and answer the question again.

(d) Increase the number of periods to $T = 500$. Plot the solution in a figure. How many periods does it take for the optimal sequence to reach the steady state? In how many periods does the optimal sequence start to decumulate capital?

6. **[Bonus (not required to submit)](Coding the Stochastic Growth Model)**. Modify the program of the deterministic growth model (with infinite time) to accommodate stochastic productivity. Suppose the production function is given by $y_t = A_t k_t^\alpha$, where the total factor productivity, A_t, follows a Markov chain with two states: $A_t \in \{1-a, 1+a\}$, and $Prob(A_{t+1} = A_t) = p$. Assume initially that $a = 0.05$ and $p = 0.8$.

   (a) Use *Value Function Iteration* and *grid search* to find the value function numerically. Show the value functions $V(k_i, A_1)$ and $V(k_i, A_2)$ on a graph.

   (b) How do the results change if: $a = 0.01$, $a = 0.1$, $p = 0.5$, and $p = 0.95$ (make one change at a time). Comment.

   (c) In what sense is there a Steady State now? How could we find it? Describe briefly (Hint: remember that under certain conditions the Markov chain has a unique stationary distribution).