Macroeconomics I
Problem Set 3

1. **(Bellman Equation of Consumption and Savings)**. Consider the standard consumption and savings problem. A consumer with $a_0 \geq 0$ receives a constant flow of $w > 0$ each period and can save at an interest rate (gross) $(1+r) \equiv R$. The budget constraint is:

$$c_t + \frac{a_{t+1}}{R} \leq a_t + w \quad t = 0, 1, \ldots, \infty,$$

and the *no-borrowing* constraint: $a_{t+1} \geq 0$. Also, suppose there exists an exogenous upper limit given by $\overline{a} > 0$. The consumer's utility is

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u(\cdot)$ is a continuous, differentiable, strictly increasing, strictly concave function with $u(0) = 0$. Use the theorems seen in class to answer the questions.

   (a) State this problem as a dynamic programming problem: describe the state, control variable, return function, and feasible set.

   (b) Write down the Bellman equation.

   (c) Does a unique function V satisfy the Bellman equation? Explain.

   (d) Is the solution of the Bellman equation V also the solution of the underlying sequential problem? Explain.

   (e) Show that the value function is increasing.

   (f) Show that the value function is concave and that the optimal policy function is continuous.

   (g) How would you construct the sequence $\{a_{t+1}\}_{t=0}^{\infty}$ for the optimal savings plan given a_0?

   (h) Use the Envelope Theorem and derive the Euler equation using the Bellman equation.

2. **(Consumption and Savings with Habit Formation)**. Consider an agent whose utility depends not only on current consumption c_t, but also on past consumption c_{t-1}:[1]

$$\sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1}),$$

where u is differentiable and concave in both arguments, and is increasing in c_t. In addition to $\beta \in (0,1)$ and $c_{-1} > 0$ given. The agent is endowed with assets $a_0 > 0$ and receives an income $y > 0$ each period. Suppose the agent can save at a gross interest rate $(1+r) = R > 1$ and that there is a borrowing limit given by $a > -\overline{A}$.

[1] Preferences with habit formation are used in finance models (e.g., Campbell and Cochrane (1999)), and in DSGE models (e.g., Christiano, Eichenbaum and Evans (2005)).

(a) Write down this problem in Dynamic Programming form: Describe the state variables, control variable, return function, and feasible set.

(b) Write down the Bellman equation.

(c) Find and interpret the Euler equation.

3. **(Search for Two Husbands with Dona Flor)**. Dona Flor lives in a world that lasts two periods: $t = 0, 1$. She is looking for a husband and has two dates scheduled: one with Vadinho at $t = 0$, and another with Teodoro at $t = 1$.[2] When meeting Vadinho, she observes the *match quality*, which is a random variable x distributed uniformly on $[0, 1]$; 1 means that Vadinho is the perfect husband for her, while 0 means he is quite ordinary. If she stays with him, Dona Flor receives utility x in both periods (and cancels the date with Teodoro).

   If she decides to leave Vadinho, she receives utility σ in period 0 and meets Teodoro at $t = 1$, where $\sigma \in (0, 1)$ is given exogenously. In the encounter with Teodoro, Dona Flor observes her *match quality* with Teodoro, given by y. Suppose that y is uniformly distributed on $[0, 1]$ and is independent of x. Again, Dona Flor can decide whether to stay with Teodoro and receive utility y, or to remain single, in which case she receives $\sigma \in (0, 1)$. Dona Flor maximizes expected utility and discounts the period with $\beta \in (0, 1)$.

   (a) Write down the Bellman equations that characterize Dona Flor's problem: What are the state variables, control variable, return function, and the feasible control set.

   (b) Find Dona Flor's decision rules.

   (c) How does the decision rule at time 0 change as σ increases? Briefly explain the intuition behind the result.

   (d) How does the decision rule at time 0 change as β increases? Briefly explain the intuition behind the result.

   (e) Suppose the following changes: $\sigma = 0$ and $\beta = 1$; x is distributed as before, but $y = \rho x + u$, $\rho \in (0, 1)$ and where u is a random variable uniformly distributed on $[-1, 1]$ (and independent of x).

      i. Show that the value of rejecting Vadinho is

$$V_0^R(x) = \frac{\rho^2}{4}x^2 + \frac{\rho}{2}x + \frac{1}{4}.$$

      ii. Explain intuitively why now V^R is increasing in x (but it wasn't before).

      iii. Show that the probability that Dona Flor rejects Vadinho is increasing in ρ.

      iv. For $\rho = 0.5$, find the decision rule at period $t = 0$.

4. **(Coding the Growth Model in Finite Time)**. Consider the "standard" neoclassical growth model in finite horizon. The production function k_t^α, the capital motion law is given by:

$$k_{t+1} = k_t(1 - \delta) + k_t^\alpha - c_t,$$

   and the representative household chooses the consumption sequence to maximize the following utility function:

[2] In Jorge Amado's original novel (1966), Vadinho is dead, and Dona Flor has to choose between Teodoro and Vadinho's spirit (!).

$$\sum_{t=0}^{T} \beta^t \log(c_t).$$

The Bellman equation of the problem is:

$$V_t(k) = \max_{k'}\{\log(k^\alpha + k(1-\delta) - k') + \beta V_{t+1}(k')\} \quad t = 0, ..., T-1$$

and

$$V_T(k') = \max_{k'}\{\log(k^\alpha + k(1-\delta) - k'\},$$

with the associated policy function: $k' = g_t(k)$.

(a) Carefully describe the algorithm for finding the value function and the policy function.

(b) Consider $T = 50$, $\alpha = 0.3$, $\beta = 0.96$, and $\delta = 0.1$. Discretize the capital space into $n_k = 200$ equidistant points, with $k_{min} = 2k_{ss}/n_k$ and $k_{max} = 2k_{ss}$ (k_{ss} is the steady-state capital in an infinite horizon problem). Implement the algorithm in a programming language of your choice.

(c) Set $k_0 = k_{min}$. Use the policy function to simulate the optimal capital sequence $\{k^*_{t+1}\}_{t=0}^{T}$. Represent the solution in a figure. How long does it take for the optimal sequence to reach the steady state (if it does)? Change the parameters $\beta = 0.8$ and $\beta = 0.99$ and answer the question again.

(d) Increase the number of periods to $T = 500$. Plot the solution in a figure. How many periods does it take for the optimal sequence to reach the steady state? In how many periods does the optimal sequence start to decumulate capital?