Macroeconomics I
Problem Set 2

1. **(An Economy with Two Sectors)**. Consider the following economy. The representative household has utility function:

$$U = \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(c_t), \quad \rho > 0.$$

Where $u(.)$ is strictly increasing, strictly concave, differentiable, bounded with $\lim_{c \to 0} u'(c) = \infty$. The household is endowed with one unit of time.

The economy is composed of two sectors, one for the production of consumption goods and another for the production of investment goods. The technology of the consumption goods sector is

$$c_t = A k_{c_t}^{\theta} n_{c_t}^{1-\theta}, \quad 0 < \theta < 1.$$

The technology of the investment goods sector is

$$x_t = B k_{x_t}^{\phi} n_{x_t}^{1-\phi}, \quad 0 < \phi < 1.$$

The capital accumulation technology is

$$k_{t+1} = k_t(1-\delta) + x_t, \quad 0 < \delta < 1,$$

and the feasibility constraints of the economy are:

$$n_{c_t} + n_{x_t} \leq 1 \quad \text{and} \quad k_{c_t} + k_{x_t} \leq k_t.$$

Suppose the household owns the capital stock and sells the services of capital and labor, whose prices in terms of the consumption good in period t are given by r_t and w_t, respectively. The price of the investment good in terms of the consumption good in period t is q_t. After receiving factor income, households decide how much to consume and invest. Households are born with initial capital k_0.

(a) Carefully define the problem of the household and the problem of the firms (in both sectors) in a sequential market environment.

(b) Define a competitive equilibrium for this economy.

(c) Define the intertemporal optimization problem solved by a benevolent central planner for this economy.

(d) Obtain and interpret the necessary first-order conditions for this problem.

(e) Specify a set of equations that characterize the equilibrium quantities in the steady state. (Hint: this system of equations involves the steady-state quantities c, x, k_c, k_x, k, n_c and n_x, and the steady-state prices r, q, and w.)

(f) Specify an algorithm to compute the quantities c, x, k_c, k_x, k, n_c and n_x, and the prices r, q, and w in the steady state. It is not necessary to solve the problem algebraically.

2. **(Baby Boom Dynamics).** Consider the standard Ramsey-Cass-Koopmans model with population growth (but no technological growth). The preference of the representative household is given by:

$$\max_{c_t \geq 0} \int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\sigma}}{1-\sigma} dt, \tag{1}$$

where c_t is per capita consumption. The resource constraint (per capita) of the economy is given by:

$$f(k_t) = c_t + i_t,$$
$$\dot{k}_t = i_t - (n+\delta)k_t.$$

Suppose $\rho > n$ for all changes in n in the questions.

(a) Characterize the planner's solution for this economy.

(b) Derive the equations characterizing the steady state and draw the phase diagram.

(c) Consider a permanent and unannounced increase in the population growth rate of the economy. Suppose the economy is in steady state at the time of the increase and show the dynamics in the phase diagram.

(d) Now consider a *temporary* and unannounced increase in the growth rate at t_0. That is, $n_t = n' > n$ for $t_0 \leq t \leq T$, but $n_t = n$ for $t > T$. Suppose the economy is in steady state at t_0 and show the dynamics of consumption and investment in the phase diagram and over time (*Hint*: consumption jumps only when the change in n is announced).

(e) Now consider a permanent increase in the growth rate at T, *but announced at* $t_0 < T$. Suppose the economy is in steady state at t_0. Show the dynamics of consumption and investment in the phase diagram and over time.

3. **(Elastic Labor Supply in the Balanced-Growth Path)**. Consider the neoclassical growth model in discrete time. The representative household derives utility from leisure (or alternatively disutility of work), that is, in addition to the choice of consumption and capital accumulation, the household chooses the amount of hours worked h_t. The utility is:

$$\sum_{t=0}^{\infty} \beta^t \left(\log(C_t) - \theta \frac{h_t^{1+\phi}}{1+\phi} \right),$$

where $0 < \beta < 1$. Let's assume total depreciation of capital $\delta = 1$ (for simplicity). The economy's resource constraint is:

$$Y_t = K_t^{\alpha}(A_t h_t)^{1-\alpha} = K_{t+1} + C_t,$$

where technological evolution $A_{t+1}/A_t = (1+g)$. There is no population growth, $L_0 = 1$, and $A_0 = 1$. $K_0 > 0$ is given. When solving the problem, DO NOT transform the variables into efficient units (that is, work with C_t, Y_t, and K_t, the variables that are NOT stationary in the balanced-growth path).

(a) Define and solve the social planner's problem of the economy. Write down a system of three equations that together with K_0 and the TVC allow us to find the optimal allocations $\{C_t, K_{t+1}, h_t\}_{t=0}^{\infty}$ of this economy.

(b) We know that with log utility and $\delta = 1$ we can find the agent's decision rule in closed-form form. Suppose the saving rate s is constant over time: $C_t = (1-s)Y_t$. Use the method of undetermined coefficients to show that $s = \alpha\beta$ (*Hint*: use the euler equation and the resource constraint).

(c) Show that the hours worked, h_t, is a function of the parameters and does not vary over time: $h_t = h^*$.

(d) Finally, transform the variables into efficient units of labor: $k_t = K_t/A_t$, $c_t = C_t/A_t$, and $y_t = Y_t/A_t$. Find the steady-state capital, k_{ss}, as a function of the parameters and h^*. What is the growth rate of K_t in the balanced-growth path (that is, when $k_t = k_{ss}$)?

(e) For labor hours, h_t, to be constant along the balanced-growth path it is necessary for utility to have the form:

$$u(c, l) = \frac{(cv(l))^{1-\sigma} - 1}{1-\sigma},$$

where $l = 1 - h$ is leisure and $v()$ a function.[1] Explain intuitively (without mathematical arguments) why h is constant in the long run, even when C_t, Y_t, and K_t have positive growth rates (*Hint*: think about what should happen with the income effect, the substitution effect, and complementarity of consumption and leisure for the amount of demanded leisure to remain constant).

[1] In our case, h_t is also constant outside of steady state. This is due to the assumptions of log and $\delta = 1$. In general, this does not happen.