Macroeconomics I
Problem Set 1

1. **(Solow Model Basics).** Suppose a Solow model in discrete time. The population grows at rate n, while technology, A_t, grows at rate g. Let the $w = \partial F(K, AL)/\partial L$ and $r = \partial F(K, AL)/\partial K$.

   (a) Show that the marginal product of labor, $w = A[f(\tilde{k}) - \tilde{k} f'(\tilde{k})]$, where the variables with tildes are per-capita and per efficient labor unit variables.

   (b) Show that, if F has constant returns to scale, we have $wL + rK = F(K, AL)$.

   (c) Suppose the production function is a Cobb-Douglas: $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$. Find the steady state variables $\tilde{k}_{ss}$, $\tilde{y}_{ss}$ and $\tilde{c}_{ss}$ as function of the parameters.

   (d) Let the "golden rule" be the savings rate that maximizes consumption in the steady state. Find the golden rule of this economy.

2. **(Constant Relative Risk Aversion Utility Function).** The most commonly used instantaneous utility function in macroeconomic models is isoelastic utility (also known as CRRA utility function):

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma \geq 0 \text{ and } \sigma \neq 1.$$

   For questions (c) onwards, also consider two periods with discount $0 < \beta < 1$:

$$U(c_1, c_2) = u(c_1) + \beta u(c_2),$$

   (a) Show that if $\sigma = 1$, then $u(c) = \ln c$ (hint: use L'Hôpital's rule).

   (b) Define $\sigma(c) = -\frac{u''(c)c}{u'(c)}$ as the Arrow-Pratt coefficient of relative risk aversion. Thus $\sigma(c)$ indicates the individual's attitude towards risk. Show that σ is constant.

   (c) Let the marginal rate of substitution between two periods be:

$$MRS(c_2, c_1) = \frac{\partial U/\partial c_2}{\partial U/\partial c_1}.$$

   The function u is homothetic if $MRS(c_2, c_1) = MRS(\gamma c_2, \gamma c_1)$. Show that U is homothetic.

   (d) Define the intertemporal substitution elasticity as:[1]

$$ies(c_t, c_{t+1}) = -\frac{d(c_{t+1}/c_t)}{dMRS(c_{t+1}, c_t)} \frac{MRS(c_{t+1}, c_t)}{c_{t+1}/c_t}$$

   Show that $ies(c_1, c_2) = 1/\sigma$.[2]

[1] The intertemporal substitution elasticity is defined as the percentage change in the consumption growth rate in response to a percentage change in the real interest rate. Note that the interest rate is the intertemporal price of consumption, and that at the optimum (Euler equation) it equals MRS.

[2] This relationship between the intertemporal substitution elasticity and the coefficient of risk aversion is not general. There are preferences in which the two variables are not related (for example, *Epstein–Zin preferences*).

(e) **Bonus (not required to submit):** redo all exercises with a CARA (Constant Absolute Risk Aversion) utility: $u(c) = 1 - \exp -\sigma c$. Verify that certain conditions above are not satisfied.

3. **(Negishi's Method)**. Consider a two-agent exchange economy. The utility function for agent $i = 1, 2$ is given by:

$$U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t \log c_t^i. \tag{1}$$

In each period, agents receive endowments e_t^i determined by the following rule:

$$e_t^1 = \begin{cases} \hat{e}, & \text{if } t \text{ is even;} \\ 0, & \text{if } t \text{ is odd;} \end{cases} \qquad e_t^2 = \begin{cases} 0, & \text{if } t \text{ is even;} \\ \hat{e}, & \text{if } t \text{ is odd.} \end{cases} \tag{2}$$

The endowment can be transformed into a final consumption good at no cost: $c_t = \hat{e}$. The objective of the problem is to find the optimal allocations using the Negishi method.[3]

(a) Let α^i be the weight given by the social planner to agent i (where $\sum_{i=1}^{2} \alpha^i = 1$). Define and solve the social planner's problem for this economy. Interpret the first-order conditions. How does the planner choose to allocate the economy's resources (intertemporally and among agents)?

(b) What is the set of Pareto efficient allocations for this economy?

(c) We want to find the Pareto efficient allocation that decentralizes a competitive equilibrium. Determine that the price of a unit of consumption p_t is equal to the Lagrange multiplier of the economy's resource constraint (normalize $p_0 = 1$). Define a transfer function in terms of weights, $t^i(\alpha^i)$, where t is the transfer each agent needs to be able to purchase the efficient allocation given their endowment sequence $\{e^i\}_{t=0}^{\infty}$.

(d) Briefly and intuitively explain why the transfer that implements the competitive equilibrium is equal to $t^i(\alpha) = 0$ for all i. Find the weight α that determines the competitive equilibrium.

4. **(Neoclassical Growth Model with Exchanges in Period 0)**. Consider the neoclassical growth model. There is a representative family with unit mass. There is no population growth. The utility function is given by:

$$U(c_{t t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma},$$

with k_0 given. The final good that can be consumed or invested is produced using the following production function: $y_t = k_t^{\alpha} n_t^{1-\alpha}$. The accumulation of capital is given by the following motion law:

$$k_{t+1} = k_t(1 - \delta) + i_t.$$

[3]Hint: see section 2.2.4 of DK's notes.

(a) Describe the household and the firm problem in an Arrow-Debreu market structure (exchanges in period 0).

(b) Characterize the equilibrium: solve the problems and interpret the solutions. Write down the system of difference equations that characterize the solution of this economy (including initial and terminal conditions).

(c) Define a competitive equilibrium for this economy.