Macroeconomics I

Search and Matching: The Diamond-Mortensen-Pissarides Model

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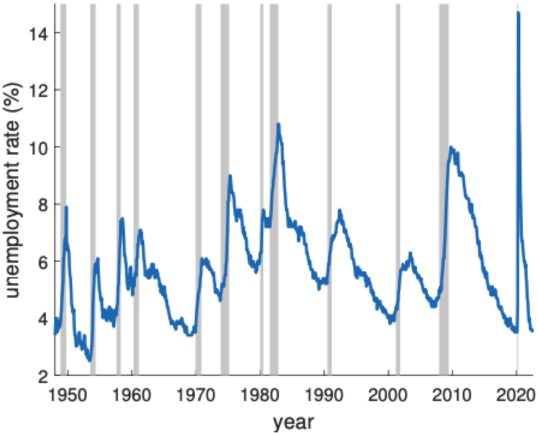
Introduction

- Thus far, we have assumed a frictionless labor market, where the supply equates the demand.
- In this case, there is no involuntary unemployment and the equilibrium wage clears the labor market.
- This is counterfactual as many individuals who want to work cannot find jobs (especially in recessions).
- In this lecture, we include search frictions in the labor market. It takes time, effort, and resources for workers and firms to match with each other.

Search Frictions

- Two main approaches use search frictions:
  - **Equilibrium wage-posting (the Burdett-Mortensen model)**: used to study wage dispersion and monopsony. The micro approach.
  - **Search-and-Matching (the Diamond-Mortensen-Pissarides model)**: used to study equilibrium unemployment. The macro approach.
- We will focus on the second case, the search-and-matching framework with random search.
- This model gave Diamond-Mortensen-Pissarides **the 2010 economics Nobel prize**.
- There is also a (kind of) third approach, which is in between the two cases called **directed/competitive search**. Definitely used more by macro researchers.

Unemployment Rate in the US



Source: CPS (via PhD Macrobook)

What We Learn in This Chapter

- How to solve the standard Diamond-Mortensen-Pissarides framework.
- When the model is efficient.
- Some puzzles and possible solutions and extension.
- How to introduce the search and matching in a RBC model.

References

- PhD Macrobook: Ch. 20.
- Rogerson, Shimer & Wright (2005): Search-Theoretic Models of the Labor Market: A Survey. *Journal of Economic Literature*.
- Pissarides (2000): Ch. 1 & 2.
- Ljungqvist, Lars and Thomas J. Sargent: Ch. 18.

Labor Market Flows

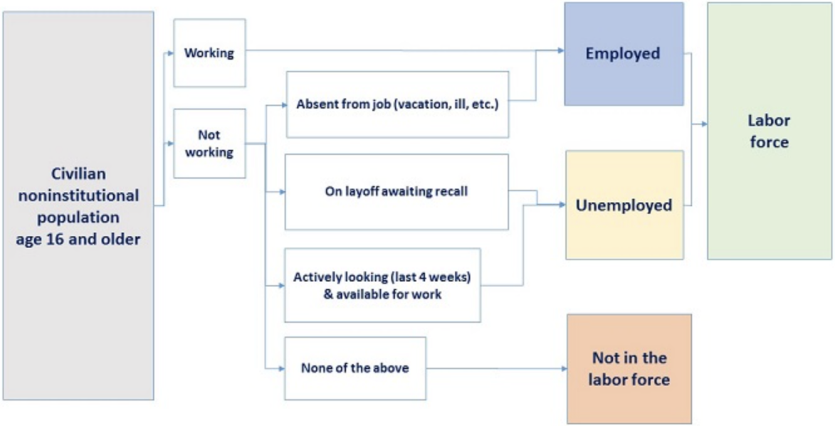
Measuring the Labor Market

General Definitions:

- **Employed:** They worked during the survey reference week.
  - It includes formal or informal workers, employees, unpaid family workers.
- **Unemployed:** Did not work during the reference week, but they looked for a job.
- **Out of the labor force:** Did not look for a job in the reference week.
  - Students, retired, stay-at-home dad/mom, but also discouraged workers.

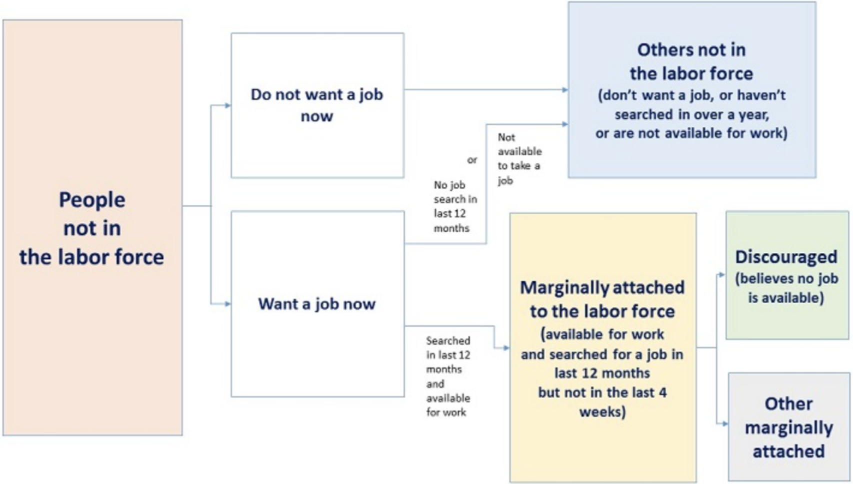
Labor force: Employed + unemployed.

CPS Definitions



Source: CPS

CPS Definitions



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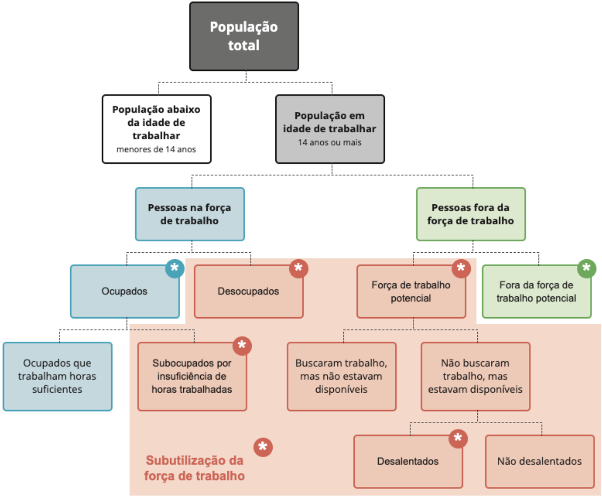
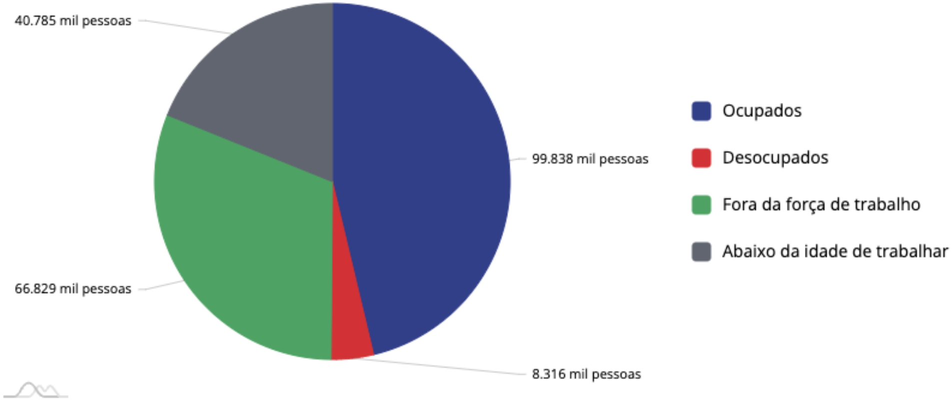
IBGE Definitions



Source: IBGE

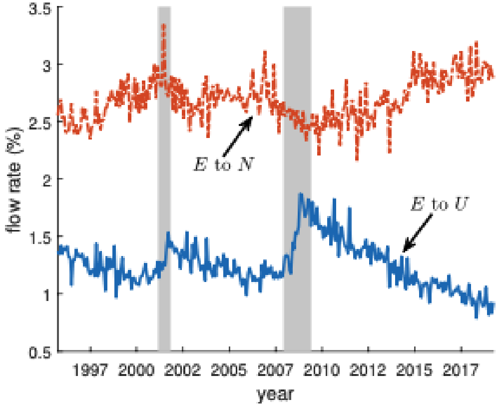
Brazil: Third Quarter 2023



Source: IBGE

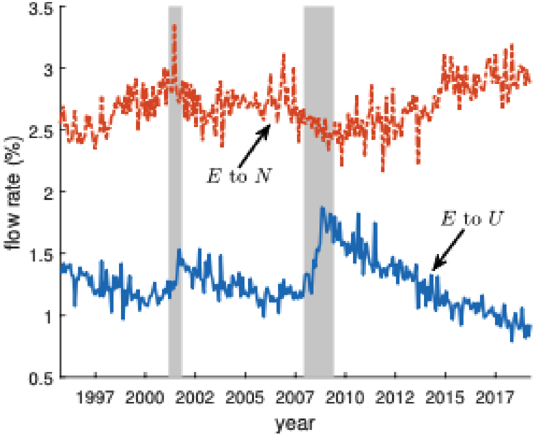
Stocks and Flows: USA

Figure: Fraction of employed workers that flows into unemployment (gross flow)



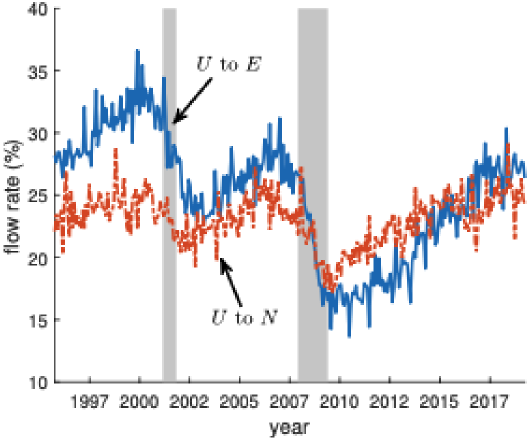
Source: Fallick and Fleischman (2004) (via PhD macrobook).

USA Stocks and Flows: E to U & N



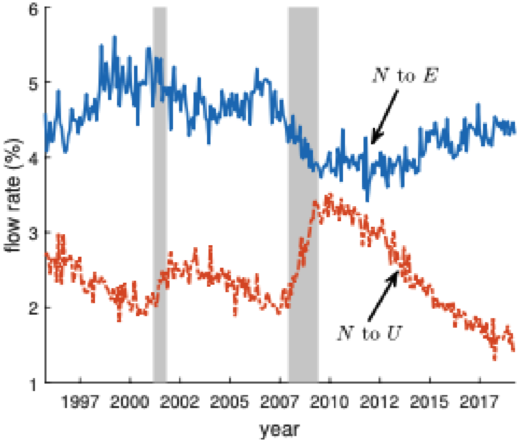
Source: Fallick and Fleischman (2004) (via PhD macrobook).

USA Stocks and Flows: U to E & N



Source: Fallick and Fleischman (2004) (via PhD macrobook).

USA Stocks and Flows: N to E & U



Source: Fallick and Fleischman (2004) (via PhD macrobook).

A Stock & Flow Unemployment Model

- Let us consider a simple unemployment model. Ignore flows-in-out of the labor force.
- Normalize the labor force to 1, so:

$$e_t + u_t = 1$$

where e_t is the employment and u_t the unemployment rate at time t.

- The law of motion of the unemployment rate is:

$$u_{t+1} = u_t(1-\lambda) + \sigma \underbrace{(1-u_t)}_{e_t}$$

where λ is the **job-finding rate** and σ the **job-separation rate**.

A Stock & Flow Unemployment Model

- In the steady state, the unemployment rate is constant, $u_{t+1} = u_t = u_{ss}$, and is:

$$u_{ss} = u_{ss}(1-\lambda) + \sigma(1-u_{ss}) \quad \Leftrightarrow \quad u_{ss} = \frac{\sigma}{\lambda + \sigma}$$

- Subtracting the LOM at t from the one at $t-1$, we can find speed of convergence to the SS:

$$u_{t+1} - u_t = \underbrace{(1-\lambda-\sigma)}_{\text{speed of convergence}} (u_t - u_{t-1})$$

- Finally, the job-finding rate is also linked to expected unemployment duration: $D = 1/\lambda$.

The Diamond-Mortensen-Pissarides Model

The Diamond-Mortensen-Pissarides (DMP) Model

- The stock and flow is useful to think about unemployment, but it does not say anything about how job-finding rate or the separation rate are defined.
- We will use **search frictions** to endogenize the job-finding rates (and later separation rates).
- Again, we assume workers are either employed or unemployed (normalize the population to one). Unemployed workers search for jobs.
- Firms search for workers by posting (costly) vacancies. Once an unemployed worker match with a firm, the firm produces y and pays a wage w.
- Search is **random**: all vacancies have the same chance of finding workers, and all workers have the same chance of finding the vacancies.

The Matching Function

- The number of matches depends on how many vacancies and unemployed workers are in the economy. We summarize it through a **matching function**:

  $$\mathcal{M}_{t+1} = M(u_t, v_t)$$

  where u_t and v_t are the number of unemployment and vacancies at t, and $\mathcal{M}_{t+1}$ is the number of matches in the next period.

- We assume that M is increasing in both arguments, concave, and homogenous of degree 1 (i.e., CRS).

- The **matching function** is a black box. It summarizes the complex problem of recruiting activities, search costs, etc, but it is analytically convenient and it has been estimated for many countries.

The Matching Function

- Since search is random, the probability of a worker meeting a firm (i.e., job-finding rate) is:

$$\frac{M(u_t, v_t)}{u_t} = M\left(1, \frac{v_t}{u_t}\right) = M(1, \theta_t) \equiv \lambda_w(\theta_t)$$

where $\theta_t \equiv v_t / u_t$ is the **labor market tightness**.

- Similarly, the probability of a vacancy meeting a worker is:

$$\frac{M(u_t, v_t)}{v_t} = M\left(\frac{u_t}{v_t}, 1\right) = M\left(\frac{1}{\theta_t}, 1\right) \equiv \lambda_f(\theta_t)$$

note that $\lambda_w(\theta_t) = \theta_t \lambda_f(\theta_t)$

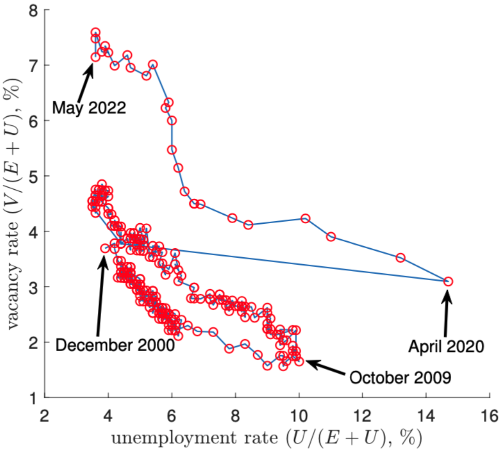
Beveridge Curve

- The job-finding and vacancy filling probability depend on the number of traders.
- The higher is the unemployment, the lower is the probability the worker finds a job $\Rightarrow$ **Congestion externalities**.
- Substituting the job-finding probability in the SS unemployment rate:

$$u_{ss} = \frac{\sigma}{\lambda_w(v/u_{ss}) + \sigma} \quad \Leftrightarrow \quad M(v, u_{ss}) + \sigma u_{ss} = \sigma$$

- There is a **negative relation between unemployment rate and vacancy rate** $\Rightarrow$ This is known as the **Beveridge curve**.

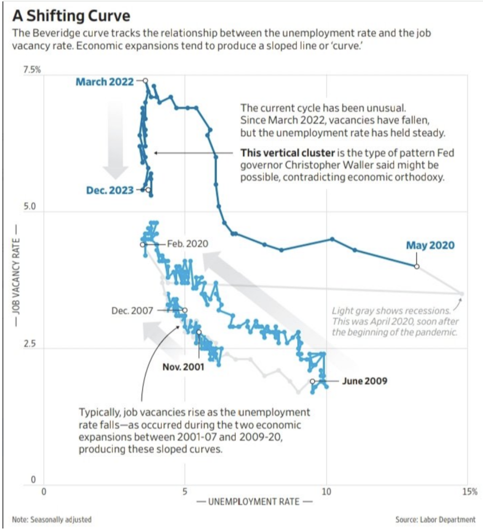
Beveridge Curve: US



Source: CPS and JOLTS (via PhD macrobook).

Why the Beveridge Curve is shifting? See Barlevy et al (2024, JEP).

Beveridge Curve



Source: The Wall Street Journal

Matched Firms

- What determine the labor market tightness?
- For simplicity, imagine that a firm hires only one worker. Ignore capital.
- Assume a match between one firm and one worker produce z_t goods, where z_t follows a 1st order Markov process.
- Let the value of matched firm be:

$$J(z) = z - w(z) + \beta \mathbb{E}\left[(1-\sigma) J(z') + \sigma V(z') \middle| \, z \right]$$

where $w(z)$ is the wage paid to the worker, $z - w(z)$ is the flow profit, $\beta \in (0,1)$ and $V(z')$ the value of a unmatched firm with open vacancy.

Open Vacancy

- The Bellman equation of a firm with open vacancy is:

$$V(z) = -\kappa + \beta \mathbb{E}\left[\lambda_f(\theta) J(z') + (1 - \lambda_f(\theta)) V(z') \middle| \, z\right]$$

  where κ is the cost of posting a vacancy. Some models assume it depends on z.

- Anyone can set up a vacancy. Thus, if the value of opening a vacancy is > 0 more firms will do it. But more vacancies reduce the probability the vacancy will be filled.

- In equilibrium, the value of opening a vacancy is driven down to zero. This is the **free entry condition**:

$$V(z) = 0 \qquad \Rightarrow \qquad \frac{\kappa}{\lambda_f(\theta)} = \beta \mathbb{E}[J(z')|z] \tag{1}$$

Job Creation Condition

- Using the free entry condition in the Bellman eq. of a matched firm:

$$J(z) = z - w(z) + \beta(1-\sigma)\mathbb{E}\left[J(z')\middle|\, z\right] = z - w(z) + (1-\sigma)\frac{\kappa}{\lambda_f(\theta)}$$

- Using Equation (1) and substituting $J(z')$ using the equation above:

$$\begin{aligned}\frac{\kappa}{\lambda_f(\theta)} =& \beta\mathbb{E}[J(z')|z] \\ \frac{\kappa}{\lambda_f(\theta)} =& \beta\mathbb{E}\left[z' - w(z') + (1-\sigma)\frac{\kappa}{\lambda_f(\theta')}|z\right]\end{aligned}$$

- This is the **intertemporal job creation condition**. What is the intuition of this equation?

Digression: Sequential Problem

- To gather intuition, let's solve the problem again using the sequential approach.
- Suppose there is a representative firm that is deciding how many vacancies to post in order to maximize the sum of discount profits. The sequential problem is:

$$
\begin{aligned}
\max_{e_{t+1}, v_t} \quad & \mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta^t (z_t e_t - w_t e_t - \kappa v_t) \right] \\
\text{s.t.} \quad & e_{t+1} = (1-\sigma) e_t + \lambda_f(\theta_t) v_t
\end{aligned}
$$

- Note that we are using the employment law of motion (instead of unemployment). The representative firm chooses employment in $t+1$ by posting more vacancies in t (taking as given θ_t).

Digression: Sequential Problem

- Let μ_t be the Lagrange multiplier. The Lagragean is:

$$\mathcal{L} = \mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta^t (z_t e_t - w_t e_t - \kappa v_t) + \mu_t((1-\sigma)e_t + \lambda_f(\theta_t)v_t - e_{t+1}) \right]$$

and the f.o.c. with respect to v_t and e_{t+1}

$$\beta^t \frac{\kappa}{\lambda_f(\theta_t)} = \mu_t \qquad \text{and} \qquad \mu_t = \mathbb{E}_t \left[\beta^{t+1}(z_{t+1} - w_{t+1}) + \mu_{t+1}(1-\sigma) \right]$$

which implies the **job creation condition**:

$$\frac{\kappa}{\lambda_f(\theta_t)} = \beta \mathbb{E}_t \left[z_{t+1} - w_{t+1} + (1-\sigma) \frac{\kappa}{\lambda_f(\theta_{t+1})} \right]$$

Job Creation Condition

$$\frac{\kappa}{\lambda_f(\theta)} = \beta \mathbb{E}\left[z' - w(z') + (1-\sigma)\frac{\kappa}{\lambda_f(\theta')} | z \right]$$

- **Intuition**:
  - **The LHS**: marginal cost of hiring an additional worker (cost of vacancy discounted by the vacancy filling probability).
  - **The RHS**: discounted marginal benefit of hiring an additional worker, the net profit in t $(z' - w(z'))$ plus the continuation value if the match survives (which is equal of the mg. value of hiring in $t+1$).
- Lower wage $\rightarrow$ higher value of a job $\rightarrow$ more vacancies $\rightarrow$ higher tightness.
- To close the model, we must determine wages.

Workers

- Assume infinitely-lived workers, with linear utility that do not save:

$$E_0\left[\sum_{t=0}^{\infty}\beta^t c_t\right]$$

- Workers earn the wage w when employed and b when unemployed (b is the outside option, unemployment benefits or home production).
- The Bellman Equations for employed ,$W(z)$, and unemployed workers, $U(z)$, are:

$$\begin{aligned}W(z) &= w(z) + \beta\mathbb{E}[(1-\sigma)W(z') + \sigma U(z')|z],\\ U(z) &= b + \beta\mathbb{E}[\lambda_w(\theta)W(z') + (1-\lambda_w(\theta))U(z')|z].\end{aligned}$$

Matching Protocol

- Workers always accept the match as long as $W > U$. Firms always accept the match as long as $J > V = 0$.
- In every period, the match generates z, if they separate the worker generates b and the firm nothing. As long as $z > b$, the match generates positive surplus.
- How can we split the surplus through wages? Any wage $w \in (b, z)$ is accepted by both the firm and the worker.
- We assume that the firm and the worker split the surplus following the **Generalized Nash Bargaining protocol**.

Nash Bargaining

- The Nash Bargaining splits the surplus so that the weighted product of surpluses of each party is maximized. The problem solves:

$$\max_{w} \left(W(z;w) - U(z)\right)^{\gamma} \left(J(z;w) - V(z)\right)^{1-\gamma}$$

  where γ is the weight of the worker (i.e., her bargaining power).

- The f.o.c gives:

$$(1-\gamma)\left(W(z;w) - U(z)\right) = \gamma\left(J(z;w) - V(z)\right) \qquad (2)$$

- Note that $V(z) = 0$ by the FE condition.

Wage Equation: Math

- Multiply the Equation $J(z)$ by γ:

$$\gamma J(z) = \gamma(z - w) + \beta\gamma(1-\sigma)\mathbb{E}\left[J(z')\middle| z\right]$$

- Subtract $W(z)$ by $U(z)$ and multiply by $(1-\gamma)$

$$(1-\gamma)\left(W(z) - U(z)\right) = (1-\gamma)(w-b) + \dots$$
$$\dots\beta\mathbb{E}[(1-\sigma)(1-\gamma)(W(z') - U(z')) - \lambda_w(\theta)(1-\gamma)(W(z') - U(z'))|z]$$

- Combining the previous two equations with Equation (2):

$$\gamma(z-w) + \beta\gamma(1-\sigma)\mathbb{E}\left[J(z')\middle| z\right] = (1-\gamma)(w-b) + \dots$$
$$\dots\beta\mathbb{E}[(1-\sigma)(1-\gamma)(W(z') - U(z')) - \lambda_w(\theta)(1-\gamma)(W(z') - U(z'))|z]$$

Wage Equation: Math

- Re-arranging:

$$w = (1-\gamma)b + \gamma z + \dots$$

$$\beta \mathbb{E}\left[(1-\sigma)\underbrace{(\gamma J(z') - (1-\gamma)(W(z') - U(z'))}_{=0 \text{ by Eq. (2)}} + (1-\gamma)\lambda_w(\theta)(W(z') - U(z'))\right]$$

- Thus, the **wage equation**:

$$w = b + \gamma(z-b) + \beta(1-\gamma)\lambda_w(\theta)\mathbb{E}\left[W(z') - U(z')|z\right] \tag{3}$$

- Depends on:
  - the share of production surplus: $b + \gamma(z-b)$;
  - the opportunity cost of looking for a job, that depends on the $\lambda_w(\theta)$ and the future economic conditions.

Wage Equation

- Substituting (2) in the FE condition, (1), then:

$$\frac{\kappa}{\lambda_f(\theta)} = \beta\mathbb{E}[J(z')|z] = \frac{\beta(1-\gamma)}{\gamma}\mathbb{E}\left[W(z) - U(z)|z\right]$$

- Thus, we can write the **wage equation** as:

$$w = b + \gamma(z - b) + \beta(1-\gamma)\lambda_w(\theta)\mathbb{E}\left[W(z') - U(z')|z\right]$$
$$w = b + \gamma(z - b) + \gamma\kappa\theta$$

- So the wage only depends on fluctuations in the productivity, labor market tightness and parameters.

Equilibrium

- The model equilibrium is summarized by three equations:

$$\frac{\kappa}{\lambda_f(\theta_t)} = \beta \mathbb{E}_t \left[z_{t+1} - w_{t+1} + (1-\sigma) \frac{\kappa}{\lambda_f(\theta_{t+1})} \right] \qquad \text{(Job Creation)}$$

$$w_t = b + \gamma (z_t - b) + \gamma \kappa \theta_t \qquad \text{(Wage Equation)}$$

$$u_{t+1} = u_t (1 - \lambda_w(\theta_t)) + \sigma (1 - u_t) \qquad \text{(Unemployment LOM)}$$

- Substituting the wage equation in the job creation:

$$\frac{\kappa}{(1-\gamma)\lambda_f(\theta_t)} = \beta \mathbb{E}_t \left[z_{t+1} - b + \frac{1 - \sigma - \gamma \lambda_w(\theta_{t+1})}{1-\gamma} \frac{\kappa}{\lambda_f(\theta_{t+1})} \right]$$

Equilibrium: Steady State

- In the Steady State, we must determine v_{ss} and u_{ss}. The job creation in the steady state,

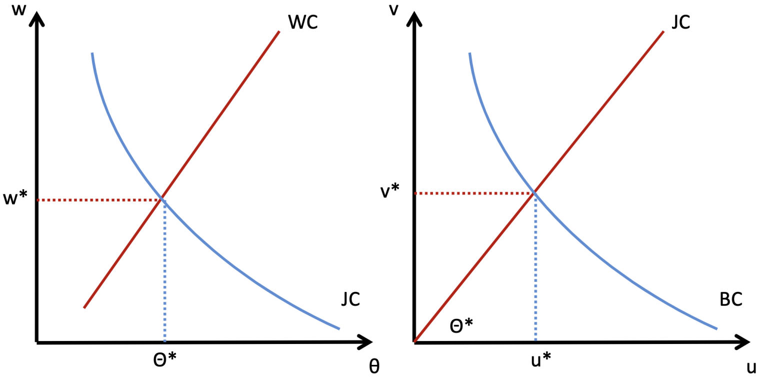
$$\frac{\kappa}{(1-\gamma)\lambda_f(\theta_{ss})} = \beta\left[z_{ss} - b + \frac{1-\sigma-\gamma\lambda_w(\theta_{ss})}{1-\gamma}\frac{\kappa}{\lambda_f(\theta_{ss})}\right],$$

together with the Beveridge curve,

$$u_{ss} = \frac{\sigma}{\lambda_w(v_{ss}/u_{ss})+\sigma},$$

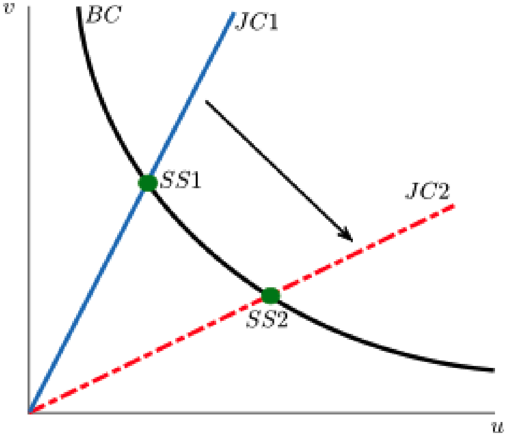
determines the steady state value of v_{ss} and u_{ss} (and $\theta_{ss} \equiv v_{ss}/u_{ss}$).

Equilibrium in the Steady State



- Wage curve (WC) depends positively on θ.
- Job creation (JC) and Beveridge curve (BC) pin down vacancies and unemployment.

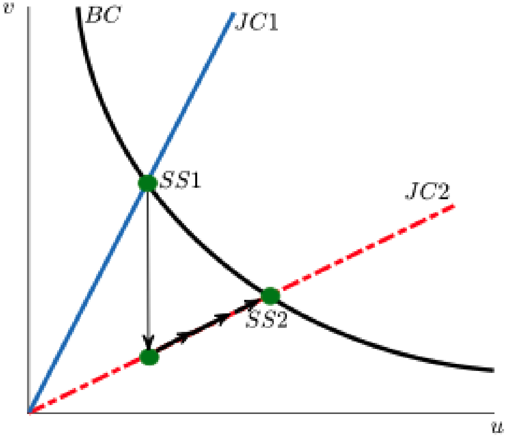
Steady State: Permanent Fall in z



- JC represents the number of v as a function of u.
- When z falls, firms have less incentives to create vacancies.
- The line becomes flatter $\rightarrow$ higher u and lower v.

Source: PhD Macrobook.

Transition Dynamics: Permanent Fall in z



Source: PhD Macrobook.

- The transition to the new SS **is slow**.
- A fall in z changes the JC immediately, but u respond slowly.
- v **jumps down** and u follows the LOM until it reaches the new SS.
- During the transition, the equilibrium points are NOT in the Beveridge curve.

Quantitative Evaluation

- How well the model can explain the worker flows across employment and unemployment?
  - The flow rate from U to E is procyclical, the model is qualitatively consistent with that.
  - The flow rate from E to U is countercyclical, the model cannot account for that.

- To evaluate the quantitative performance, we must choose functional forms. The matching function is Cobb-Douglas:

$$M(u,v) = \chi u^{\eta} v^{1-\eta}, \quad \text{then} \quad \lambda_w(\theta) = \chi \theta^{1-\eta} \quad \text{and} \quad \lambda_f(\theta) = \chi \theta^{-\eta}$$

  where $\eta \in (0,1)$.

- Productivity follows an AR(1): $\ln(z_{t+1}) = (1-\rho)\ln(z_{ss}) + \rho \ln(z_t) + \sigma_{\varepsilon}\varepsilon_{t+1}$, where $\varepsilon \sim N(0,1)$.

Digression: Numerical Solutions

- The model can be solved numerically using standard techniques:
  - Log-linearization (i.e., perturbation), value function iteration / projection, shooting algorithm, etc.

- For more intuition, see the log-linearized solution in the PhD macrobook.

- If you are using linearized solutions, you have to be careful about corner solutions (i.e., the number of vacancies hits zero, $v_t = 0$).

- For more information, see Petrosky-Nadeau & Zhang (QE, 2017): *Solving the Diamond-Mortensen-Pissarides Model Accurately*.

Calibration: Shimer (2005)

- Most of the parameters are taken from Shimer (AER, 2005). The shock process comes from Hagedorn and Manovskii (AER, 2008).

Calibrated Parameters	Value
β	0.996
ρ	0.949
σ_ε	0.0065
σ	0.034
χ	0.45
b	0.4
γ	0.72
η	0.72

Source: PhD Macrobook.

Quantitative Evaluation: The Shimer Puzzle

- The model generates the right correlations, but the magnitude of the fluctuations in u, v, and θ is too small.
- This is known as **labor market volatility puzzle** (or Shimer puzzle (2005)).
- The left is the data, the right table is model moments:

		u	v	v/u	z
Standard Deviation		0.125	0.139	0.259	0.013
Quarterly Autocorrelation		0.870	0.904	0.896	0.765
	u	1	−0.919	−0.977	−0.732
Correlation Matrix	v	—	1	0.982	0.460
	v/u	—	—	1	0.967
	z	—	—	—	1

		u	v	v/u	z
Standard Deviation		0.005	0.016	0.020	0.013
Quarterly Autocorrelation		0.826	0.700	0.764	0.765
	u	1	−0.839	−0.904	−0.804
Correlation Matrix	v	—	1	0.991	0.972
	v/u	—	—	1	0.961
	z	—	—	—	1

Solution 1: Wage Rigidity

- One reason why the elasticities of u and v to z are small is because the wages increase too much in booms, so it weakens the response of profit to a shock in z.

- A solution to the puzzle is **wage rigidity**. Instead of Nash Bargaining, assume wage is fixed at w_{ss}:

$$\frac{\kappa}{\lambda_f(\theta_t)} = \beta\mathbb{E}_t\left[z_{t+1} - w_{ss} + (1-\sigma)\frac{\kappa}{\lambda_f(\theta_{t+1})}\right]$$

		u	v	v/u	z
Standard Deviation		0.115	0.329	0.425	0.013
Quarterly Autocorrelation		0.825	0.693	0.763	0.765
Correlation Matrix	u	1	−0.791	−0.881	−0.784
	v	—	1	0.986	0.969
	v/u	—	—	1	0.961
	z	—	—	—	1

- Another alternative, wages partially rigid (Hall, 2005): $w = \alpha w^{NB} + (1-\alpha)w_{ss}$ (w^{NB} is the wage from Nash Bargaining).

Solution 2: Hagedorn-Manovskii Calibration

- Hagedorn and Manovskii (2008) propose to re-calibrate the outside option b and the bargaining weight γ.
  - (i) First, they add elastic vacancy costs.
  - (ii) Second, they choose very high b so that is very close to z, and very low γ.
- By reducing the surplus (particularly from the workers), an increase in z has a huge impact on profits and hence in vacancies.
- But it implies that people are almost indifferent between working and being unemployed. Is it realistic?

Efficiency & Extensions

Endogenous Separation

- The flow rate of $E \to U$ increases in recessions, but in the model it is constant. One extension to account for this fact is to include **endogenous separation**.

- Suppose the firm has to pay a cost for maintaining the match, $c(\sigma)$, where $c'(\sigma) < 0$.

- Now σ is a choice variable, so the Bellman equation of a matched firm is:

$$J(z) = \max_{\sigma} z - w(z) - c(\sigma) + \beta \mathbb{E}\left[(1-\sigma) J(z') + \sigma V(z') \middle| z \right]$$

- The f.o.c is (using the FE condition in the second equality):

$$-c'(\sigma) = \beta \mathbb{E}\left[J(z') \middle| z \right] = \frac{\kappa}{\lambda_f(\theta)}$$

Endogenous Separation

- The optimal value of σ will be a function of z, so it will fluctuate when z fluctuates.
- Assume: $c(\sigma) = \phi\sigma^{-\xi}$. The model generates more fluctuation than the constant σ, but it is still very small.

		u	v	v/u	z
Standard Deviation		0.010	0.011	0.021	0.013
Quarterly Autocorrelation		0.862	0.623	0.764	0.765
	u	1	−0.893	−0.969	−0.909
Correlation Matrix	v	—	1	0.976	0.957
	v/u	—	—	1	0.961
	z	—	—	—	1

- One alternative is to include wage rigidity.

Efficiency

- Because there is search frictions, the decisions of the agents - firms posting vacancies and unemployed workers accepting matches - impact the other firms and workers.
- For instance, a new vacancy increases marginally the probability of a match, but it also decreases the probability of all the other vacancies to fill.
- This is the **congestion externality**.
- To have a sense when the model is efficient, we must solve the Social Planner's problem.
- The Planner is still subject to the search frictions, hence the solution is **constrained efficient**.

Planner's Problem

- The social planner maximizes the total surplus:

$$\max_{e_{t+1},v_t} \quad \mathbb{E}_t\left[\sum_{t=0}^{\infty}\beta^t(z_t e_t - b(1-e_t) - \kappa v_t)\right]$$
$$\text{s.t.} \quad e_{t+1} = (1-\sigma)e_t + \lambda_f(\theta_t)v_t$$

- The Planner explicitly considers the effect of v_t and e_t on $\theta_t \equiv \frac{v_t}{1-e_t}$!!!

- Let μ_t be the Lagrange multiplier. The Lagragean is:

$$\mathcal{L} = \mathbb{E}_t\left[\sum_{t=0}^{\infty}\beta^t(e_t(z_t-b) - b - \kappa v_t) + \mu_t((1-\sigma)e_t + \lambda_f\left(\frac{v_t}{1-e_t}\right)v_t - e_{t+1})\right]$$

Planner's Problem

- The f.o.c. with respect to e_{t+1}:

$$\mu_t = \mathbb{E}_t \left[\beta^{t+1}(z_{t+1} - b) + \mu_{t+1}\left((1-\sigma) + \lambda_f'(\theta_{t+1})\theta_{t+1}^2\right)\right]$$

- The f.o.c with respect to v_t:

$$\beta^t \frac{\kappa}{\underbrace{\left(\lambda_f'(\theta_t)\theta_t + \lambda_f(\theta_f)\right)}_{=\lambda_f(\theta_t)[1-\eta(\theta_t)]}} = \mu_t$$

  where $\eta(\theta_t) \equiv -\frac{\theta_t \lambda_f'(\theta_t)}{\lambda_f(\theta_t)}$ is the elasticity of the vacancy filling probability, i.e., how much the probability of filling a vacancy changes with a change in the tightness.

- The focs include new terms accounting for the externality.

- In the Cobb-Douglas case: $\eta(\theta_t) = \eta$.

Planner's Problem

- Combining both equations, using $\lambda_f'(\theta)\theta \equiv -\eta(\theta)\lambda_f(\theta)$, and simplifying stuff:

$$\beta^t \frac{\kappa}{\lambda_f(\theta_t)[1-\eta(\theta_t)]} = \mathbb{E}_t \left[\beta^{t+1}(z_{t+1} - b) + \beta^t \frac{\kappa}{\lambda_f(\theta_{t+1})[1-\eta(\theta_{t+1})]} \left((1-\sigma) + \lambda_f'(\theta_{t+1})\theta^2_{t+}\right.\right.$$

$$\frac{\kappa}{\lambda_f(\theta_t)[1-\eta(\theta_t)]} = \beta\mathbb{E}_t \left[z_{t+1} - b + \frac{\kappa(1-\sigma)}{\lambda_f(\theta_{t+1})[1-\eta(\theta_{t+1})]} - \kappa\theta_{t+1}\frac{\eta(\theta_{t+1})}{1-\eta(\theta_{t+1})}\right]$$

- Ok, now what? Let's look at the Job Creation equation derived using the market equilibrium:

$$\frac{\kappa}{(1-\gamma)\lambda_f(\theta_t)} = \beta\mathbb{E}_t \left[z_{t+1} - b + \frac{\kappa(1-\sigma)}{(1-\gamma)\lambda_f(\theta_{t+1})} - \kappa\theta_{t+1}\frac{\gamma}{(1-\gamma)}\right]$$

Hosios Condition

- The Planner solution is equal to the market equilibrium whenever: $\eta(\theta) = \gamma$! This is called **Hosios condition** (Hosios, 1990).
- The share of workers (firms) in the surplus of a match is equal to the elasticity of the matching function with respect to the corresponding search input.
- Intuitively, we want to tax the "search" to correct for the negative externality.
- The bargaining weight acts as a distortionary tax here: once you have a match the firm/worker only appropriates part of the surplus.
    - Some people call this the **appropriability** problem
- The Hosios condition means that the appropriability and congestion exactly balance each other.

Search & Matching in the RBC

- Including a Search and Matching framework into a standard neoclassical growth model is relatively straightforward. Early papers include Merz (1995) and Andolfatto (1996).

- Suppose the representative household has employed and unemployed workers. The household utility is usual: $\mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$.

- The budget constraint considers the income of the employed and unemployed members:

$$c_t + k_{t+1} = (1 + r_t - \delta)k_t + (1 - u_t)w_t + u_t b + d_t$$

  where d_t is the profit of the firm.

- The solution implies the usual Euler Equation. In particular, define the **stochastic discount factor** as:

$$Q_{t+1} = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]$$

Search & Matching in the RBC: Firms

- The representative firm produces according to: $y_t = z_t k_t^\alpha e_t^{1-\alpha}$, where e_t is the number of individuals employed.

- The sequential problem of the firm is similar to the one presented before:

$$\max_{e_{t+1}, v_t, k_t} \quad \mathbb{E}_t \left[\sum_{t=0}^{\infty} \prod_{s=1}^{t} Q_s \underbrace{\left(z_t k_t^\alpha e_t^{1-\alpha} - r_t k_t - w_t e_t - \kappa v_t \right)}_{=d_t} \right]$$

$$\text{s.t.} \quad e_{t+1} = (1-\sigma) e_t + \lambda_f(\theta_t) v_t$$

- Note that k_t is a static decision, hence the f.o.c:

$$\alpha z_t k_t^{\alpha-1} e_t^{1-\alpha} = r_t \qquad \Rightarrow k_t = \left(\frac{\alpha z_t}{r_t} \right)^{\frac{1}{1-\alpha}} e_t$$

Search & Matching in the RBC: Firms

- Using the solution for capital:

$$d_t = z_t k_t^{\alpha} e_t^{1-\alpha} - r_t k_t - w_t e_t - \kappa v_t$$

$$d_t = (1-\alpha)\frac{y_t}{e_t} e_t - w_t e_t - \kappa v_t$$

$$d_t = \underbrace{(1-\alpha) z_t^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{r_t}\right)^{\frac{\alpha}{1-\alpha}}}_{=mpn(z_t, r_t)} e_t - w_t e_t - \kappa v_t$$

- Thus, the problem is almost the same as we had before:

$$\max_{e_{t+1}, v_t} \quad \mathbb{E}_t \left[\sum_{t=0}^{\infty} \prod_{s=1}^{t} Q_s (mpn(z_t, r_t) e_t - w_t e_t - \kappa v_t) \right]$$

$$\text{s.t.} \quad e_{t+1} = (1-\sigma) e_t + \lambda_f(\theta_t) v_t$$

Search & Matching in the RBC: Job Creation

- The first order conditions imply the usual job creation equation that depends on the stochastic discount factor and the interest rate:

$$\frac{\kappa}{\lambda_f(\theta_t)} = \mathbb{E}_t \left[Q_{t+1} \left(mpn(z_{t+1}, r_{t+1}) - w_{t+1} + (1-\sigma) \frac{\kappa}{\lambda_f(\theta_{t+1})} \right) \right]$$

- Since the unemployment law of motion is the same, it remains to show the wage determination to derive the rest of the model.

- We will show that the marginal value of employment in the model with capital has a close connection with the worker equations seen before.

Search & Matching in the RBC: Employment Value

- We could derive the marginal value of employment in the sequential problem of the household:

$$\mathcal{L} = \mathbb{E}_t[\sum_{t=0}^{\infty} \beta^t u(e_t w_t + (1-e_t)b + d_t + (1+r_t-\delta)k_t - k_{t+1}) + ...$$
$$...\mu_t^e \left(e_t(1-\sigma) + (1-e_{t-1})\lambda_w(\theta_{t-1}) - e_t\right)]$$

where μ_t^e is the multiplier of the employment law of motion.

- The f.o.c. w.r.t to e_t:

$$\mu_t^e = (w_t - b)\beta^t u'(c_t) + \mathbb{E}_t[\mu_{t+1}^e \left(1 - \sigma - \lambda_w(\theta_t)\right)]$$

Search & Matching in the RBC: Employment Value

- Re-define the multiplier: $\mu_t^e = \hat{\mu}_t^e \beta^t u'(c_t)$. Then, the previous equation:

$$\hat{\mu}_t^e = w_t - b + \mathbb{E}_t \left[\underbrace{\frac{\beta u'(c_t)}{u'(c_{t+1})}}_{Q_{t+1}} [\hat{\mu}_{t+1}^e \left(1 - \sigma - \lambda_w(\theta_t)\right)] \right]$$

- Remind you of something? Recall $W_t - U_t$ in the case without capital:

$$W_t - U_t = w_t - b + \beta \mathbb{E}_t \left[\left(W_{t+1} - U_{t+1}\right) \left(1 - \sigma - \lambda_w(\theta_t)\right) \right]$$

- The marginal value of employment $\hat{\mu}_t^e \equiv W_t - U_t$ if we weight for the fact that utility is concave and there are savings.

Search & Matching in the RBC: Wage Equation

- Using the marginal value of employment and the marginal value of a job filled (this is also a multiplier in the firm problem) in the **Nash Bargaining**, we find the wage equation:

$$w_t = b + \gamma(mpn(z_t, r_t) - b) + \gamma\kappa\theta_t$$

  which is exactly the same as before - except that the marginal product of labor depends on r as well.

- Together with the **Job Creation Condition** and **employment law of motion**, we have the search and matching block of the model:

$$\frac{\kappa}{\lambda_f(\theta_t)} = \mathbb{E}_t\left[Q_{t+1}\left(mpn(z_{t+1}, r_{t+1}) - w_{t+1} + (1-\sigma)\frac{\kappa}{\lambda_f(\theta_{t+1})}\right)\right]$$

Search & Matching in the RBC: Equilibrium

- The rest of the model are the usual equations from the RBC/Neoclassical growth model:

$$
\begin{aligned}
&u'(c_t) = \beta \mathbb{E}_t[(1 + r_{t+1} - \delta)u'(c_{t+1})]) \\
&z_t k_t^{\alpha} e_t^{1-\alpha} = c_t + k_{t+1} - (1 - \delta)k_t + \kappa v_t \\
&r_t = \alpha z_t (e_t/k_t)^{1-\alpha} \\
&\ln(z_{t+1}) = (1 - \rho)\ln(z_{ss}) + \rho \ln(z_t) + \sigma_{\varepsilon}\varepsilon_{t+1}
\end{aligned}
$$

- Note we must consider the vacancy cost in the resource constraint.

Quantitative Evaluation

		u	v	v/u	z
Standard Deviation		0.005	0.017	0.022	0.015
Quarterly Autocorrelation		0.819	0.688	0.755	0.763
	u	1	−0.831	−0.899	0.089
Correlation Matrix	v	—	1	0.991	−0.071
	v/u	—	—	1	−0.078
	z	—	—	—	1

	Y	C	I	L	Y/L
Standard Deviation	0.014	0.003	0.059	0.0004	0.014
Correlation with Y	1	0.875	0.991	0.902	0.99992

Source: PhD Macrobook.

- As in the baseline model, unemployment volatility is very low. Wage rigidity fixes this.