Macroeconomics I

Dynamic Programming

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Introduction

- Up to this point, we have solved dynamic problems by finding the optimal sequence.
- This is not always practical and often counterintuitive.
- From now on, we will study how to solve the problem recursively, exploiting the fact that decisions can be made period by period.
- This method is known as **Dynamic Programming**.
- It is particularly useful for solving problems numerically.

What We Learn in This Chapter

- The mathematical foundations of Dynamic Programming.
- How to write dynamic problems recursively.
- How to solve Dynamic Programming problems through value function iteration.
- How to write a recursive competitive equilibrium.
- The basics of markov chains and how to include it in DP problems.

References

- Dirk Krueger's notes: Ch. 3, 4 and 5.
- Acemoglu: Ch. 6, and 16.
- PhD Macrobook Ch. 4.
- Pretty much the entire Stokey and Lucas with Prescott.

A Simple Example

- A simple example using the neoclassical growth model.
- We study the general case in detail later.

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \tag{1}$$

$$\text{s.t.} \quad 0 \leq k_{t+1} \leq f(k_t) + (1-\delta)k_t \quad \text{for all } t \tag{2}$$

$$k_0 \text{ given.} \tag{3}$$

- Note that $V(k_0)$ is the total value of the problem at time 0 for an economy starting with capital k_0.

A Simple Example

Suppose $\delta = 1$ for simplicity.

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \tag{4}$$

$$= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ u(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right\} \tag{5}$$

$$= \max_{\{k_1\}} \left\{ u(f(k_0) - k_1) + \beta \left[\max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right] \right\} \tag{6}$$

$$= \max_{\{k_1\}} \left\{ u(f(k_0) - k_1) + \beta \underbrace{\left[\max_{\{k_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^{t} u(f(k_{t+1}) - k_{t+2}) \right]}_{=V(k_1)} \right\} \tag{7}$$

$$V(k_0) = \max_{k_1} u(f(k_0) - k_1) + \beta V(k_1) \tag{8}$$

A Simple Example

$$V(k_0) = \max_{0 \leq k_1 \leq f(k_0) + (1-\delta)k_0} u(f(k_0) + (1-\delta)k_0 - k_1) + \beta V(k_1) \tag{9}$$

- Instead of maximizing an infinite sequence, we only need to find k_1.
- On the other hand, we do not know the form of the $V()$ function, i.e., $V()$ is a *functional equation*.
- Note that the solution $k_1 = g(k_0)$ is a function of k_0.

Bellman Equation

- Since the problem is the same in all periods, we can generalize:

$$V(k) = \max_{0 \leq k' \leq f(k)+(1-\delta)k} u(f(k) + (1-\delta)k - k') + \beta V(k') \tag{10}$$

  where k is current capital and k' is capital in the next period.

- This is the famous value function or the **Bellman Equation**.
- Under what conditions can we generalize? Conceptually, the two problems are different:
  - $V(k_0)$ is the sequential formulation, the value of the discounted infinite sum of the utility evaluated at the optimum.
  - $V(k)$ is the recursive problem, the value function that solves the dynamic programming problem.
- Under certain conditions, the solutions to these two problems are the same.

Bellman Equation

- If the solutions to these two problems are the same, there is no need to find the entire sequence of $\{k_t\}_{t=0}^{\infty}$ to solve the model!
- We can just find the solution (i.e., the $\max$) of the recursive problem:

$$V(k) = \max_{0 \le k' \le f(k)} \{u(f(k) - k') + \beta V(k')\} \tag{11}$$

  which is the function $k' = g(k)$!
- Then starting from k_0 we can find the entire sequence: $k_1 = g(k_0)$, $k_2 = g(k_1)$, ..., $k_{t+1} = g(k_t)$!
- **Problem:** we cannot solve the $\max$ since we do know $V(k')$.

Mathematical Preliminaries

Bellman Equation

Functional Equation:

$$V(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \tag{12}$$

- x is the state variable.
- y is the control variable.
- $\Gamma : X \to Y$ is the feasible set correspondence.
- $F : X \times Y \to \mathbb{R}$ is the instantaneous return function.

$$g(x) = \arg \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \tag{13}$$

- $g(x)$ is the policy function.

Bellman Equation

Sequential Problem:

$$V^*(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad \text{(SP)} \tag{14}$$

$$\text{s.t.} \quad x_{t+1} \in \Gamma(x_t) \quad \text{for all } t \tag{15}$$

$$x_0 \text{ given.} \tag{16}$$

Functional Equation:

$$V(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \quad \text{(FE)} \tag{17}$$

- Under which conditions is the solution to the SP problem equal to the FE?
- How can we find the solution to the FE problem, and under what conditions is the solution unique?

The Operator T

- To solve the sup and find the policy function $g(x)$, we first need to find the function V.
- Define the T **operator**:

$$(TV)(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \tag{18}$$

- The T operator is a "function' that maps a function V to another function V, i.e., $T : C \to C$, where C is the set of possible functions.
- The ultimate goal is to find the **fixed point** of the operator, i.e., find the function V for which $V = TV$.
- We will use the **Banach Fixed-Point Theorem**.

Banach Fixed-Point Theorem

Theorem (Banach Fixed-Point Theorem - Contraction Mapping Theorem)

If (S, d) is a complete metric space and $T : S \rightarrow S$ is a contraction with modulus β, then:

1. *T has exactly one fixed point in S, i.e., there exists only one V such that $TV = V$;*
2. *For any $v_0 \in S$, $d(T^n V_0, V) \leq \beta^n d(V_0, V)$, for $n = 1, 2, ..$*

- Writing the operator as:

$$V_{n+1}(x) = (TV)(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V_n(y)\} \tag{19}$$

- If the conditions of Banach are satisfied, the theorem gives us a simple algorithm: guess V_0 and iterate on the operator until the distance between V_n and V_{n+1} is sufficiently small.
- It also guarantees the uniqueness of V!
- **Proof:** (SLP/A) Use the definition of contraction and the triangle inequality property of the norm.

Road Map

- We need to define the domain of the T operator and what convergence of sequences means within this space $\Rightarrow$ Define a complete metric space.

- Then we have to define what a contraction is and under what conditions T is a contraction.

- Sometimes it's not trivial to show that T is a map from a function to itself (especially when there is a $\sup$). We will use the Berge's Maximum Theorem to guarantee this.

- Finally, knowing that we are in a complete metric space and that T is a contraction and maps a function to itself, we can approximate our value function using the **Banach Fixed-Point Theorem**.

Metric Spaces

Definition (Vector Space)

A vector space X is a set that is closed under vector addition (finite) and scalar multiplication. Let $f,\ g \in X$ and $\alpha \in \mathbb{R}$:

1. Addition: $(f+g)(x) = f(x) + g(x)$
2. Scalar multiplication: $(\alpha f)(x) = \alpha f(x)$

To discuss convergence, we also need a notion of distance (between two elements within a set):

Definition (Metric Space)

A metric space is a non-empty set S and a metric $d : S \times S \to \mathbb{R}$ such that for all $x, y, z \in S$:

1. $d(x, y) \geq 0$ with equality if $x = y$;
2. $d(y, x) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Metric Spaces

- For vector spaces, we define metrics in such a way that the distance between two vectors is equal to the distance between their difference and zero: $d(x, y) = d(x - y, \vec{0})$.

Definition (Normed Vector Space)

A normed vector space is a vector space S and a norm $||.|| : S \to \mathbb{R}$ such that for all $x,\ y \in S$ and $\alpha \in \mathbb{R}$:

1. $||x|| \geq 0$ with equality if and only if $x = \vec{0}$;
2. $||\alpha x|| = |\alpha|\ ||x||$;
3. $||x + y|| \leq ||x|| + ||y||$.

- In other words, a normed vector space is a pair $(X, ||.||)$, where X is a vector space and $d(x, y) = ||x - y||$.
- Okay, but we are interested in the distance between functions.

Metric Spaces with Functions

- **Example:** Let $C(X)$ be the set of continuous and bounded functions with domain $[a, b]$ in $\mathbb{R}$, and $x,\ y \in C(X)$. Define $d(x, y)$ as:

$$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|. \tag{20}$$

  Thus, the pair $(C(X), d)$ is a metric space.
- Verify that the conditions are satisfied:
  1. $d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)| = |x(t^*) - y(t^*)| \geq 0$ where t^* is the maximizer, and equality if and only if $x = y$;
  2. $d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)| = \max_{t \in [a,b]} |y(t) - x(t)| = d(y, x)$.
  3. $d(x, z) = \max_{t \in [a,b]} |x(t) - z(t)| = |x(t^*) - z(t^*)| \leq |x(t^*) - y(t^*)| + |y(t^*) - z(t^*)| = \max_{t \in [a,b]} |x(t) - y(t)| + \max_{t \in [a,b]} |y(t) - z(t)| = d(x, y) + d(y, z)$
- In general, we will use the supremum norm (uniform norm) as the measure of distance between functions: $||f|| = \sup_{x \in X} |f(x)|$.

Convergence of Sequences

- We have a definition of space and distance. Now we can define a sequence convergence applied to any metric space.

Definition (Convergence of Sequences)

A sequence $\{x_n\}_{n=0}^{\infty}$ in S converges to $x \in S$ if, for every $\varepsilon > 0$, there exists an N_ε such that:

$$d(x_n, x) < \varepsilon, \text{ for all } n \geq N_\varepsilon \tag{21}$$

- In other words, a sequence $\{x_n\}_{n=0}^{\infty}$ in a metric space (S, d) if and only if the sequence $\{d(x_n, x)\}_{n=0}^{\infty}$ converges to zero.

Definition (Cauchy Sequence)

A sequence $\{x_n\}_{n=0}^{\infty}$ in S is a Cauchy sequence if, for every $\varepsilon > 0$, there exists an N_ε such that:

$$d(x_n, x_m) < \varepsilon, \text{ for all } n, \ m \geq N_\varepsilon \tag{22}$$

- **Note:** Every convergent sequence is Cauchy, but the converse is not true.

Convergence of Sequences

- Intuitively, to determine if a sequence is Cauchy, it is enough to know the points of the sequence and not necessarily where it converges.
- This makes it easier to identify a Cauchy sequence than a convergent sequence.

Definition (Complete Metric Space)

A metric space (S, d) is complete if every Cauchy sequence in S converges to an element in S.

- Ok, but we are interested in the convergence of $\lim_{n\to\infty} V_n = V$, and now?

Banach Space

Theorem

Let $X \subseteq \mathbb{R}^l$ and $C(X)$ be the set of continuous and bounded functions $f : X \to \mathbb{R}$ with the supremum norm, $||f|| = \sup_{x \in X} |f(x)|$. Then $C(X)$ is a complete normed space (Banach Space).

- **Proof (intuition):** It is necessary to demonstrate that $C(X)$ is a normed space and, more importantly, complete. This involves showing that there exists a Cauchy sequence f_n. The trick is that convergence in the supremum norm is uniform convergence, and uniform convergence preserves continuity.
- In other words, we have a sequence of functions V_n in $C(X)$ and the limit of the sequence is also in $C(X)$.
- Now that we know we are looking for our function V in a Banach space, if the operator T is a contraction, we can use the Banach Fixed-Point Theorem.

Contraction

Definition (Contraction)

Let (S, d) be a metric space, and $T : S \to S$ be a function that maps S to itself. T is a contraction with modulus β if for some $\beta \in (0, 1)$, $d(Tx, Ty) \leq \beta d(x, y)$, for all $x,\ y \in S$.

- Okay, satisfying the Banach Space is easy: just choose continuous and bounded functions and use the supremum norm. How to show that T is a contraction?
- Use the definition of contraction and check if it is satisfied by T.
- Often it is complicated, and that's why it is convenient to use the **Blackwell's Sufficient Conditions**.

Contraction

Theorem (Blackwell's Sufficient Conditions)

Let $X \subseteq \mathbb{R}^l$, and let $B(X)$ be the space of bounded functions: $f : X \to \mathbb{R}$, with the supremum norm. Let $T : B(X) \to B(X)$ be an operator that satisfies:

1. *(monotonicity) $f,\ g \in B(X)$ and $f(x) \leq g(x)$ for all $x \in X$ implies that $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$;*
2. *(discount) There exists some $\beta \in (0, 1)$ such that $[T(f + c)(x) \leq (Tf)(x) + \beta c$ for all $f \in B(X)$, $c \geq 0$, and $x \in X$.*

Then T is a contraction with modulus β.

Example

Neoclassical Growth Model:

$$(TV)(k) = \max_{0 \le k' \le f(k)} \{u(f(k) - k') + \beta V(k')\} \tag{23}$$

1. (monotonicity) Let $W(k) \ge V(k)$ for all k.

$$(TW)(k) = \max_{0 \le k' \le f(k)} u(f(k) - k') + \beta W(k') \tag{24}$$

$$\ge \max_{0 \le k' \le f(k)} u(f(k) - k') + \beta V(k') = (TV)(k) \tag{25}$$

   for a $0 \le k' \le f(k)$ (fixed k, i.e., the possible set does not change), and $W(k') \ge V(k')$ by assumption.

2. (discount) For $c \ge 0$:

$$[T(V + c)](k) = \max_{0 \le k' \le f(k)} \{u(f(k) - k') + \beta(V(k') + c)\} \tag{26}$$

$$= \max_{0 \le k' \le f(k)} \{u(f(k) - k') + \beta V(k')\} + \beta c = (TV)(k) + \beta c \tag{27}$$

Maximum Theorem

- Note that for T to be a contraction, the operator needs to result in a function within the same space $C(X)$.
- Under normal conditions, it is easy to demonstrate this (sum of continuous bounded functions is continuous and bounded, etc.), but in our case, we have the $\sup$ that makes the situation more complicated.
- Consider the problem:

$$\sup_{y \in \Gamma(x)} f(x, y) \tag{28}$$

- Suppose $f(x, .)$ is continuous in y (for a fixed x) and $\Gamma(x)$ is a compact and non-empty set. Hence, the maximum exists, and the value function is well-defined:

$$h(x) = \max_{y \in \Gamma(x)} f(x, y), \tag{29}$$

as well as the optimal policy correspondence:

$$G(x) = \arg\max_{y \in \Gamma(x)} f(x, y) = \{y \in \Gamma(x); f(x, y) = h(x)\} \tag{30}$$

Maximum Theorem

Theorem (Bergé's Maximum Theorem)

Let $f : X \times Y \to \mathbb{R}$ be a continuous function, and $\Gamma : X \to Y$ be a non-empty, continuous, and compact-valued correspondence. Then:

1. *The value function $h : X \to \mathbb{R}$ is continuous;*
2. *The decision rule $G : X \to Y$ is non-empty, upper hemi-continuous, and has compact values.*

Lemma (Convex Maximum Theorem)

Let $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^m$. Suppose $\Gamma : X \to Y$ is non-empty, continuous, has compact and convex values. Let $f : X \times Y \to \mathbb{R}$ be a continuous and concave function, for each $x \in X$.

1. *The value function $h(x)$ is concave, and the correspondence $G(x)$ has convex values.*
2. *If f is strictly concave in y for every x, then $G(x)$ is continuous with a unique value (not a correspondence).*

Maximum Theorem

- Note that the Maximum Theorem ensures that our operator has a solution and that the solution is continuous.
- There is also a generalized version where f is a correspondence, but it will not be necessary for our problems.
- The lemma ensures that the solution is unique and that the operator has a concave solution.

Example: Neoclassical Growth Model

- $u(f(k) - k') + \beta V(k')$: u and f are continuous functions, so if $V(k')$ is continuous, the sum will be a continuous function.
- $\Gamma(k) = [0, f(k)]$: 0 and $f(k)$ are continuous functions of k, so $\Gamma(k)$ is non-empty, continuous, and has compact values.

By the **Maximum Theorem**, $V(k) = \max_{k' \in [0, f(k)]} u(f(k) - k') + \beta V(k')$ is also continuous. With similar arguments, we can say that $V(k)$ is bounded and concave.

Dynamic Programming

- That is, in the neoclassical growth model (and in many others), we can guess a solution $V_0(k)$ that is continuous and bounded (and depending on the problem, concave).

- Given the usual assumptions on u, f, and $\beta \in (0, 1)$, we can establish via the Maximum Theorem and Blackwell's Sufficient Conditions that the operator $(TV)(k)$ is a contraction.

- Since our distance metric between functions is the supremum norm, we are in a Banach space and can apply the Banach Fixed-Point Theorem.

- Therefore, there is only one solution V, and we can approximate it by iterating via the operator, $V_{n+1} = TV$, until the point where the distance between $||V_{n+1} - V_n||$ is small enough.

Dynamic Programming Under Certainty

Bellman Equation

Sequential Problem:

$$V^*(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad \text{(SP)} \tag{31}$$

$$\text{s.t.} \quad x_{t+1} \in \Gamma(x_t) \quad \text{for all } t \tag{32}$$

$$x_0 \text{ given.} \tag{33}$$

Functional Equation:

$$V(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \quad \text{(FE)} \tag{34}$$

Questions:

1. Does the solution of (FE) also satisfy (SP)? Is the policy function equivalent to the optimal sequence?
2. How can we find the solution to (FE)?

Notation and Definitions

- X is the set of possible values of the state variable.
- **Possible Plan**: a sequence $\{x_{t+1}\}_{t=0}^{\infty}$ satisfying $x_{t+1} \in \Gamma(x_t)$ for all t.
- A set of possible plans (given x_0): $\Pi(x_0) = \{\{x_{t+1}\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t)\}$.
- For every $n = 0, 1...$, the partial sum of (discounted) returns given a possible plan $\tilde{x}$ is defined as:

$$u_n(\tilde{x}) = \sum_{t=0}^{n} F(x_t, x_{t+1}). \tag{35}$$

Principle of Optimality

- The idea that (FE) $\Leftrightarrow$ (SP) is called the Principle of Optimality.

- Basically, there are four steps:
  1. Show that the supremum of (SP) $V^*(x_0)$ satisfies (FE): $(SP) \Rightarrow (FE)$.
  2. Show that if there exists a solution to (FE) (and if $\lim_{n\to\infty} \beta^n V(x_n) = 0$), then it is given by $V^*(x_0)$: $(FE) \Rightarrow (SP)$.
  3. Show that the sequence $\{x_{t+1}\}_{t=0}^{\infty}$ that achieves the supremum of (SP) satisfies $V = V^*$.
  4. Show that *any* sequence $\{x_{t+1}\}_{t=0}^{\infty}$ that satisfies $V = V^*$ and $\lim_{n\to\infty} \beta^n V(x_n) \leq 0$ achieves the supremum of (SP).

- Assumptions:
  - **(A1)** $\Gamma(x)$ is non-empty.
  - **(A2)** $\lim_{n\to\infty} u_n(\tilde{x})$ exists for every $\tilde{x} \in \Pi(x_0)$ (a sufficient condition is to have $F(x_t, x_{t+1})$ bounded and $\beta \in (0,1)$).

- That's it. Pretty simple, right?

Principle of Optimality

- We won't go through the complete proof (see SLP Theorems 4.2-4.5), but instead, we'll provide some intuition.
- First: **(A1)** and **(A2)** ensure that (SP) is uniquely well-defined.
- **(A1)** is not very interesting, just ensuring that we can choose some sequence.
- **(A2)** is where all the power of the **Principle of Optimality** is, along with the assumption $\lim_{n\to\infty} \beta^n V(x_n) = 0$.
  - Note that (FE) can have multiple solutions. Remember the conditions of the Contraction Mapping Theorem.
  - But if there is a solution to (FE), and the solution satisfies the extra condition $\lim_{n\to\infty} \beta^n V(x_n) = 0$, then this is the solution to (SP) (which is necessarily unique).
  - Remember the TVC. There is no equivalent condition for the recursive form, but in a sense, the extra condition puts an upper limit on utility growth.
  - SLP has some interesting examples to illustrate this condition.

Bellman Equations

- Now, we've already established that under quite mild assumptions, (SP) $\Leftrightarrow$ (FE).
- We can then focus on Bellman's equations and study this problem more carefully, including how to find the solution.
- Assumptions (let's differentiate from the previous ones):
  - **(B1)**: $\Gamma(x)$ is a non-empty, continuous correspondence with compact values.
  - **(B2)**: $F(x, y)$ is bounded and continuous.
- **(Thm)** Suppose **(B1)** and **(B2)**. We can use the mathematical tools from the last section and:
  - Define an operator $T : C(X) \rightarrow C(X)$.
  - T: has exactly one unique fixed point.
  - For any $V_0 \in C(X)$, we can approximate via iteration $|T^n V_0 - V|| \leq \beta^n ||V_0 - V||$, $n = 0, 1, 2...$
  - The policy correspondence G is upper hemicontinuous and has compact values.

Bellman Equations

- More assumptions:
  - **(B3)**: $F(x, y)$ is strictly concave.
  - **(B4)**: $\Gamma(x)$ has convex values.

- **(Thm)** Suppose **(B1)**, **(B2)**, **(B3)**, and **(B4)**: V is strictly concave, and G is continuous and uniquely defined. In other words, G is a policy function.

- Here we use the lemma of the Maximum Theorem with convexity.

- Note that the neoclassical growth model trivially satisfies these assumptions.

- But it is not uncommon to find models that do not satisfy these assumptions (e.g., models with discrete choice, where the individual chooses to work or not, etc.)

Bellman Equations

- More assumptions:
  - **(B5)**: For every y, $F(.,y)$ is strictly increasing.
  - **(B6)**: $\Gamma(x)$ is monotone. In other words, if $x \leq x'$, then $\Gamma(x) \subseteq \Gamma(x')$.
- **(Thm)** Suppose **(B1)**, **(B2)**, **(B5)**, and **(B6)**: V is strictly increasing.
- Sketch of the proof. Let $x_0 < x_1$:

$$
\begin{aligned}
V(x_0) &= \max_{y \in \Gamma(x_0)} \{F(x_0, y) + \beta V(y)\} \\
&= F(x_0, g(x_0)) + \beta V(g(x_0)), \text{ for some } g(x_0) \\
&< F(x_1, g(x_0)) + \beta V(g(x_0)) \\
&\leq \max_{y \in \Gamma(x_1)} \{F(x_1, y)) + \beta V(y)\} = V(x_1)
\end{aligned}
$$

- **Example:** Show that the neoclassical growth model satisfies **(B5)**, and **(B6)**.
- Monotonicity of the value function is a property often exploited numerically to find the solution to Bellman's equation.

Bellman Equations

- Finally, it is interesting to think about how to use calculus to characterize the solution of (FE).
- We saw that under certain conditions, the Euler Equation is a necessary condition (but not sufficient - remember TVC) for a solution.
- **(B7)**: F is continuously differentiable in the interior of the set $X \times Y$.
- How can we know the result of the differentiation on V? **Envelope Theorem**.

Theorem (Benveniste-Scheinkman or Envelope Theorem)

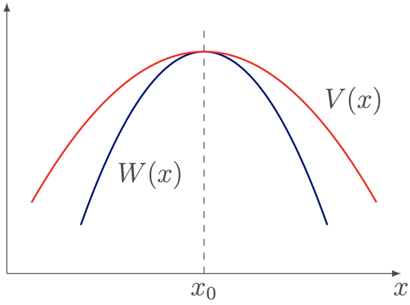
Let $X \subseteq \mathbb{R}^l$, $V : X \to \mathbb{R}$ be concave, $x_0 \in \text{int}(X)$, and D a neighborhood of x_0. If there exists a differentiable function $W : X \to \mathbb{R}$ with $W(x_0)$ and $W(x) \leq V(x)$ for every $x \in D$, then V is differentiable at x_0, and $V_i(x_0) = W_i(x_0)$ for $i = 1, 2, ..., l$.

Envelope Theorem

- The envelope theorem tells us that if we find a function $W(x) \leq V(x)$, we can use the derivative of this function to find $V_i(x)$.
- In our case:

$$\begin{aligned} W(x) =& F(x, g(x_0)) + \beta V(g(x_0)) \\ \leq& \max_y \{F(x, y) + \beta V(y)) = V(x) \end{aligned}$$

- Note that $g(x_0)$ is the optimal policy at x_0 (but may not be at x) and $V(g(x_0))$ is a number (not a function).
- Therefore: $W_i(x_0) = F_i(x_0, g(x_0)) = V_i(x_0)$.



Envelope Theorem

- **(Thm)** Suppose **(B1)**, **(B2)**, **(B3)**, **(B4)**, and **(B7)**. If $x_0 \in \text{int}(X)$ and $g(x_0) \in \text{int}(\Gamma(x_0))$, then V is continuously differentiable at x_0, where the derivative is given by:

$$V_i(x_0) = F_i(x_0, g(x_0)), \quad i = 1,\ 2, ..., l.$$

- In words: the derivative of the value function is equal to the derivative of the return function, $F(x,y)$, at the arguments x with y evaluated at the optimum.

- In the neoclassical growth model (with $\delta = 1$): $V_k(k_0) = u'(f(k_0) - g(k_0))f'(k_0)$.

- Intuitively: $V_k(k_0) = u'()f'(k_0) \underbrace{-g'(k_0)u'() + g'(k_0)\beta V'(g(k_0))}_{=0 \text{ f.o.c. (interior)}}$

Euler Equation

- Given our assumptions, we can derive an Euler Equation for the problem:

$$V(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta V(y)\}$$
$$g(x) = \arg\max_{y \in \Gamma(x)} \{F(x,y) + \beta V(y)\}$$

- F.O.C. (interior solution of $\max$): $F_y(x, y^*(x)) + \beta V_y(y^*(x)) = 0$.
- Applying the Envelope Theorem: $V_x(x) = F_x(x, y^*(x))$.
- Substituting, we find the general form of the Euler Equation:

$$F_y(x, y^*(x)) + \beta V_y(y^*(x)) = 0$$
$$F_y(x, y^*(x)) + \beta F_x(y^*(x), y^*(y^*(x))) = 0$$
$$\text{or } F_{x_{t+1}}(x_t, x_{t+1}) + \beta F_{x_{t+1}}(x_{t+1}, x_{t+2}) = 0$$

Example

Once again, let's look at the Neoclassical Growth Model (with $\delta = 1$)

$$V(k) = \max_{k' \in [0, f(k)]} u(f(k) - k') + \beta V(k') \tag{36}$$

- State variable: k;
- Control variable: k';
- Feasible set: $\Gamma(k) = [0, f(k)]$;
- Return function: $F(k, k') = u(f(k) - k')$.

Assumptions

- $f : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing, and concave;
- $f(0) = 0$ and for some $\bar{k} > 0$, $k \leq f(k) \leq \bar{k}$, for every $k \in [0, \bar{k}]$ and $f(k) < k$ for every $k > \bar{k}$.
- $u : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing, and concave; $\beta \in (0, 1)$.

Example

- It is easy to show that most of the required assumptions are satisfied.
- Perhaps the less intuitive one is that the return function should be bounded.
- Just remember that given the assumptions on the production function, the economy will eventually reach the steady state.
  - If we start the economy at $k_0 < k_{ss}$, there will be capital accumulation until k_{ss}. Therefore, capital will always be bounded by k_{ss}.
  - If $k_0 > k_{ss}$, there will be capital decumulation until k_{ss}. Therefore, capital will be bounded by k_0.
- Capital is limited by $\max\{k_0, k_{ss}\}$.
- Therefore, $\Gamma(k)$ has compact value, and $u(k, k')$ is bounded. Given the other assumptions, **(B1)** and **(B2)** are satisfied. $\Rightarrow$ the Principle Optimality and Banach Fixed Point can be applied.

Example

- u is strictly concave, and clearly $\Gamma \in [0, f(k)]$ has convex value (since $f(k)$ is continuous), so **(B3)** and **(B4)** are satisfied.
  - Lemma of the Maximum Theorem applies, and the value function will be strictly concave, and the optimal policy will be a continuous function.
- $u(k, k')$ is strictly increasing in k, and $\Gamma \in [0, f(k)]$ is monotonic (since $f(k)$ is strictly increasing), so **(B5)** and **(B6)** are satisfied.
  - $V(k)$ is a strictly increasing function.
- u and f are differentiable, **(B7)**, so the function is differentiable (Envelope Theorem). If k is interior:

$$V'(k) = u'(f(k) - g(k))f'(k)$$

Note that if the Inada conditions are satisfied, $g(k)$ will be interior.

Example

- Finally, we take the first-order condition of the functional equation problem:

$$u'(f(k) - g(k)) = \beta V'(k)$$

- Combining with the Envelope Theorem:

$$u'(f(k) - g(k)) = \beta f'(g(k))u'(f(g(k) - g(g(k)))$$
$$u'(c_t) = \beta f'(k_{t+1})u'(c_{t+1})$$

- Finally, we find our Euler Equation.

Finding the Value Function

How to Find the Value Function?

1. Guess and verify / method of undetermined coefficients.

2. Iterative process (e.g., value function iteration).

Guess and Verify

- Under certain conditions, we can solve the sequential problem analytically.
- Similarly, we can solve the value function analytically in special cases.
- Suppose $u(c) = \ln(c)$, $f(k) = k^\alpha$ and $\delta = 1$.
- Guess that the value function has the following form:

$$V = A + B\ln(k)$$

  A and B are the coefficients that need to be found.

- Let's proceed in 3 steps.

Guess and Verify

Step 1: Solve the maximization problem

$$V = \max_{0 \le k' \le k^\alpha} \{\ln\big(k^\alpha - k'\big) + \beta(A + B\ln\big(k'\big))\}$$

The FOC is sufficient, and the solution is interior:

$$k' = \frac{\beta B k^\alpha}{1 + \beta B}$$

Step 2: Evaluate the right-hand side at the optimal k'

$$V = -\ln(1 + \beta B) + \alpha\ln(k) + \beta A + \beta B\ln\left(\frac{\beta B}{1 + \beta B}\right) + \alpha\beta B\ln(k)$$

Guess and Verify

Step 3: Substitute the left-hand side with the guess and find A and B

$$A + B\ln(k) = -\ln(1+\beta B) + \alpha\ln(k) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) + \alpha\beta B\ln(k)$$

$$(B - \alpha(1+\beta B))\ln(k) = -A - \ln(1+\beta B) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right)$$

Note that the only way for the left-hand side (which depends on k) to be equal to the right is if $(B - \alpha(1+\beta B)) = 0$:

$$B = \frac{\alpha}{1-\alpha\beta},$$

substituting B on the right and setting it to zero:

$$A = \frac{\beta}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta)\right].$$

Iterative Proces: Value Function Iteration

- By the Banach Fixed Point Theorem, we know that if we guess a function $V_0(k) \in C(k)$ and iterate forward, our guess converges geometrically (β^n) to the unique solution V.
- Pseudo-algorithm:
  1. Choose an initial guess V_0 and a tolerance level $\varepsilon > 0$.
  2. Compute V_{n+1} using the operator:

     $$V_{n+1}(k) = \max_{0 \leq k \leq f(k)+(1-\delta)k} \{u(f(k) + (1-\delta)k - k') + \beta V_n(k)\}$$

     This involves solving the maximization problem and evaluating using k'^* optimal.
  3. Calculate $d = \sup ||V_{n+1} - V_n||$.
  4. If $d < \varepsilon$, we found the value function $V_{n+1} = V$. Otherwise, update the guess, $V_n = V_{n+1}$, and return to step 2.
- Note that we won't iterate to infinity (our life is finite). On the other hand, we have to choose a small ε for a good approximation.

Value Function Iteration

- In practice, we'll use the Value Function Iteration (VFI) on the computer.
- If you want to try iterating analytically, use the example of the method of undetermined coefficients and guess $V_0 = 0$. Check that V_{n+1} approaches the found solution.
- On the computer, we have to consider a few details:
  1. How to approximate V?
  2. How to solve the maximization problem?
- I'll describe the simplest version to solve the problem numerically: VFI with piecewise linear function discretized on a grid and using grid search for maximization.
- This method is the most robust, and we know exactly the conditions for its operation, but there are faster methods that require extra assumptions, such as using the Euler Equation or the policy function.

Value Function Iteration

1. Discretize the space k into a vector with I points between $\underline{K}$ and $\overline{K}$. Define the points on the grid as $\{K_1, K_2, ..., K_I\}$.
   - The number of points I is determined by the trade-off between speed and accuracy.
   - Points can be equidistant or, depending on the problem, in the region with higher curvature of the value function.
   - Choose $\underline{K}$ and $\overline{K}$ so that $0 < \underline{K} < k_{ss} < \overline{K}$.
2. The value function will be stored in a vector with I points: $\{V_i\}_{i=1}^{I}$. Each point of V_i is the value associated with capital k_i. Initialize the vector with your "guess" V^0.
3. Compute V_i^{n+1} using the procedure for all i (grid search):

$$V_{i,j}^{n+1} = \begin{cases} u(f(k_i) + (1-\delta)k_i - k_j) + \beta V_j^n, & \text{if } f(k_i) + (1-\delta)k_i - k_j = c_{i,j} > 0 \\ -\infty, & \text{if } f(k_i) + (1-\delta)k_i - k_j = c_{i,j} \leq 0 \end{cases}$$

$$V_i^{n+1} = \max\{V_{i,1}^{n+1}, V_{i,2}^{n+1}, ..., V_{i,I}^{n+1}\}$$

4. Calculate $d = \max_{i=1,..,I} |V_i^{n+1} - V_i^n|$. If $d < \varepsilon$, we found the value function $V_{n+1} = V$. Otherwise, update the guess, $V_n = V_{n+1}$, and return to the previous step.

Value Function Iteration

- When finished, it's good to do some diagnostics:
  - Check if the chosen limits $\underline{K}$ and $\overline{K}$ are sufficiently high so that the solution is interior.
  - Try decreasing the tolerance ε or the number of points I a bit more. If your approximation is good, V shouldn't change much.
- Maximization tends to be the most computationally costly step.
  - Often, it can be sped up by exploiting properties of V (concavity, monotonicity).
  - It can be done via grid search or using interpolation with an optimization algorithm (Newton, etc.).
- The policy function (via grid search) $g_i = j$ is a map from one grid point i to another grid point j.
  - To evaluate points "outside the grid", you have to use some kind of interpolation.

Examples

Example: Finite T

- We have studied infinite horizons problems: the problem is the same in every period.
- This is not the case in problems in finite horizons or non-stationary problems.
- Consider the "Cake-Eating" problem: the agent is born with assets a_0 and must consume the cake until his death at T.

$$V(a_0) = \max_{\{a_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t), \quad u \text{ follows the usual assumptions,} \tag{37}$$

$$\text{s.t.} \quad c_t + a_{t+1} = a_t(1+r), \quad \text{for } t = 0, 1, .., T \tag{38}$$

$$a_0 \text{ given.}$$

- In period $t = 0$, the agent will choose to save some portion of the cake (regardless of a_0). In period $t = T$, the agent will choose to consume everything (no more utility at $T + 1$).

Example: Finite T

- The optimal decision will depend on the time period (his age).
- In recursive form:

$$V_t(a) = \max_{a' \in [0, a(1+r)]} \{u(a(1+r) - a') + \beta V_{t+1}(a')\} \tag{39}$$

  - State variable: a and t;
  - Control variable: a';
  - Feasible set: $\Gamma(a) = [0, a(1+r)]$;
  - Return function: $F(a, a') = u(a(1+r) - a')$.
- The age of the agent (t) is a state variable (we can also write $V(a, t)$).

Solving the Problem

- Instead of iterating the value function until it converges, we can solve finite horizon problems by *backward induction*.
- In period T: $V_{T+1} = 0$ and $g_T(a) = 0$. Thus, $V_T(a) = u(a(1+r))$.
- From there, we can find $V_{T-1}(a)$, $V_{T-2}(a), ..., V_1(a)$.
- Problems in which the value function depends on T are considered non-stationary dynamic programming problems.
- The simplest problems are finite sequences, but it also includes problems with an infinite sequence in which some parameter or function depends on T.
  - We will not study non-stationary infinite sequence problems. With some modifications, the theorems we have seen apply to these situations as well (see Acemoglu chap. 6).

Example: 2-period McCall Search Model

- Two periods: $t = 1, 2$.
- Each period, the agent receives an *iid* wage offer w from a c.d.f $F(w)$ with support $[\underline{\omega}, \overline{\omega}]$.
- Decision:
  - If accepted (A): receives w in the current period and until the end of life.
  - If rejected (R): receives $\alpha \in (\underline{\omega}, \overline{\omega})$ in the current period and receives a new offer in the next period (if alive).
- A typical *Real Option* problem (sometimes we call the value function the *asset value equation*).
- Linear utility and $\beta \in (0, 1)$: the agent maximizes: $\mathbb{E}[y_0 + \beta y_1]$, where y_t is either α or w.
- The solution requires finding the reservation wage $w_{t,R}$. The wage at which the agent is indifferent to accepting or rejecting the job offer.

Example: 2-period McCall Search Model

- State: w and t.
- Control: discrete choice $c = \{\text{A}, \text{R}\}$, or we can represent it as an indicator function $c = \{1, 0\}$, where 1 is acceptance.
- Return function: $F(w) = \begin{cases} w \text{ if } c = \text{A}, \\ \alpha \text{ if } c = \text{R}. \end{cases}$
- Feasible set: $\Gamma = \{\text{A, R}\}$.

Solution: Period 2

- Value function: $V_2(w) = \begin{cases} w \text{ if } c = \text{A}, \\ \alpha \text{ if } c = \text{R}. \end{cases}$
- Policy function: $g_2(w) = \begin{cases} \text{A if } w \geq \alpha, \\ \text{R if } \alpha < w. \end{cases}$
- In other words, $V_2(w) = \max\{w, \alpha\}$ and reservation wage $w_{2,R} = \alpha$.

Example: 2-period McCall Search Model

Solution: Period 1

- Value function:
  - Accept (A): $V_1^A(w) = w + \beta w$,
  - Reject (R): $V_1^R(w) = \alpha + \beta \mathbb{E}[V_2(w')]$
- Policy function: $g_1(w) = \begin{cases} \text{A if } V_1^A(w) \geq V_1^R(w), \\ \text{R if } V_1^A(w) < V_1^R(w). \end{cases}$
- That is, $V_2(w) = \max\{w(1+\beta), \alpha + \beta \mathbb{E}[V_2(w)]\}$, where:

$$\mathbb{E}[V_2(w)] = \int_{\underline{\omega}}^{\overline{\omega}} V_2(w') dF(w') = \int_{\underline{\omega}}^{\overline{\omega}} \max\{w', \alpha\} dF(w')$$
$$= \int_{\underline{\omega}}^{\alpha} \alpha dF(w') + \int_{\alpha}^{\overline{\omega}} w' dF(w') = \alpha F(\alpha) + \int_{\alpha}^{\overline{\omega}} w' dF(w').$$

Example: 2-period McCall Search Model

Solution: Period 1

- Reservation wage in period 1:

$$w_{1,R}(1+\beta) = \alpha + \beta \left[\alpha F(\alpha) + \int_{\alpha}^{\overline{\omega}} w' dF(w') \right]$$

$$w_{1,R} = \frac{\alpha}{(1+\beta)} + \frac{\beta}{(1+\beta)} \left[\alpha F(\alpha) + \int_{\alpha}^{\overline{\omega}} w' dF(w') \right]$$

- That is, $V_1^A(w_{1,R}) = V_1^R(w_{1,R})$

Recursive Competitive Equilibrium

Recursive Competitive Equilibrium

- Just as we used the Social Planner, dynamic programming simplifies the dynamic problem.
- But, in the end, we are interested in is the equilibrium of the model. How to describe competitive equilibrium in recursive form?
- Let's write the Bellman equation for the agents making dynamic choices. In the Neoclassical Growth model, these are the households.
- What are the state variables? k and...?
  - And the prices? r and w...
  - Prices are functions of capital (MPK and MPL).
- But the Bellman equation represents the problem of a single household, and the decision of a single household cannot change the prices of the economy!
  - An agent (in a competitive equilibrium) takes prices as given! They are **atomistic**.

Recursive Competitive Equilibrium: *The 'big K, little k' trick*

- Differentiate the "aggregate" capital of the economy, K, from the capital of a household, k: *The 'big K, little k' trick*.
- Assumptions: a continuum of individuals i with unit measure, and all agents are symmetric:

$$K = \int_0^1 k_i di = \int_0^1 k di \tag{40}$$

  meaning, the aggregate capital is the sum of the capital of all households.
  - Note that in this case, it is trivial that $K = k$ since all agents are the same and therefore make the same decision.
  - But this distinction is important if agents are heterogeneous!
- Prices r and w are functions of **aggregated** capital (and if relevant for the problem, of aggregated labor $N = \int_0^1 n_i di = 1$).
- Households do not choose the aggregate state K, but they form expectations about its evolution.

Recursive Competitive Equilibrium: Neoclassical Growth

Firm's Problem

- The firm's problem is static.
  - If firms were heterogeneous, we would have to aggregate the demand of all firms f: $K^d = \int_{f \in F} k_f^d df$.
- In this case, we will simplify and assume a representative firm (and solve the problem using aggregated variables).

$$\max_{K,L} F(K, L) - r(K)K - w(K)L$$

- First-order conditions:

$$r(K) = F_K(K, 1) \quad \text{and} \quad w(K) = F_L(K, 1)$$

- In other words, prices are functions of aggregated states.

Recursive Competitive Equilibrium: Neoclassical Growth

Household's Problem

- State: k and K.

$$V(k, K) = \max_{c,\ k' \geq 0} \{u(c) + \beta V(k', K')\}$$
$$s.t. \quad c + k' = w(K) + (1 + r(K) - \delta)k$$
$$K' = H(K)$$

- Policy functions: $c^* = g^c(k, K)$ and $k'^* = g^k(k, K)$.
- $K' = H(K)$ is the perceived law of motion by the agent.
- Agents do not choose K but form expectations about its evolution (and therefore about future prices!).
- **Rational Expectations:** Agents are rational and "know" the model. Thus, the perceived law of motion will be equal to the true law of motion.

Recursive Competitive Equilibrium: Neoclassical Growth

- **Definition:** A recursive competitive equilibrium is a value function, V, policy functions g^k and g^c, price functions, r and w, and aggregate law of motion H, that:

  1. Given the functions, r, w, and H, the value function V is the solution to the Bellman equation of households with decision rules g^k and g^c.
  2. Prices satisfy:

  $$r(K) = F_k(K,1) \quad \text{and} \quad w(K) = F_L(K,1).$$

  3. Agent's expectations are rational (or the perceived law of motion is *consistent* with the true law of motion):

  $$H(K) = g^k(K,K).$$

  4. Market clearing for all K (note we are aggregating k_i):

  $$g^c(K,K) + g^k(K,K) = F(K,1) + (1-\delta)K.$$

Recursive Competitive Equilibrium: Neoclassical Growth

- The definition of a recursive equilibrium is usual, except for point 3.
- Point 3 explicitly states that **in equilibrium** the perceived law of motion must be equal to the realized law of motion when aggregating individual agent decisions:

$$K' = H(K) = \int_0^1 k_i'^* di = g(K, K),$$

  where, in equilibrium, we can use the trick $k = K$.

- In other words, agents have **rational expectations**: they choose $g^k(k, K)$ and expect $H(K)$. Correct expectations imply $K = k$ and prices that are consistent with the choices of households.
- This implies a **Fixed Point**: Policy functions depend on K, and K is the result of aggregating g^k.
  - In this problem, this is trivial, but with heterogeneous agents, it can be a costly part of solving the model.

Example: Externality in the Production Function

- Consider an economy with a continuum of firms indexed by $j \in [0, 1]$.
- Production function of an individual firm: $y_t = k_t^\alpha l_t^{1-\alpha} K_t^\gamma$, where:
  - k_t and l_t are the amounts of capital and labor hired by an individual firm;
  - K_t is the aggregate capital of the economy (taken as given by the individual firm);
  - and $\gamma + \alpha < 1$, $\gamma \geq 0$, and $\alpha > 0$.
  - Note that if $\gamma > 0$, there is a positive externality.
- There is a representative family with a unit mass that maximizes $\sum_{t=0}^{\infty} \beta^t U(C_t)$.
- Show that in the presence of externality, the Welfare Theorems will not be satisfied, and the competitive equilibrium will not be efficient.
  - In other words: **Planner's Solution $\neq$ Competitive Eq.**

Production Externality: Planner

- Since all firms are the same (MPK and MPL are equal), the planner will allocate the same amount of capital/labor to each firm.
- The planner also internalizes the production externality.
- Assuming $L_t = 1$ for all t. The planner's problem in recursive form:

$$V(K) = \max_{C, K' > 0} \{U(C) + \beta V(K')\}$$

$$\text{s.t.} \quad C + K' = K^{\alpha+\gamma} + (1-\delta)K$$

- In summary:

$$V(K) = \max_{K' > 0} \{U(K^{\alpha+\gamma} + (1-\delta)K - K') + \beta V(K')\}$$

Production Externality: Planner

- First-order condition:

$$-U'(K^{\alpha+\gamma} + (1-\delta)K - K') + \beta V'(K') = 0$$

- Envelope condition:

$$V'(K) = [(\alpha+\gamma)K^{\alpha+\gamma-1} + (1-\delta)]U'(K^{\alpha+\gamma} + (1-\delta)K - K')$$

- Combining the two conditions, we find the Euler equation (planner):

$$U'(\underbrace{K^{\alpha+\gamma} + (1-\delta)K - K'}_{=C(K)}) = [(\alpha+\gamma)(K')^{\alpha+\gamma-1} + (1-\delta)]\beta U'(\underbrace{(K')^{\alpha+\gamma} + (1-\delta)K' - K''}_{=C(K')})$$

Recursive Equilibrium: Production Externality

- To solve for the recursive equilibrium, we need to find the supply/demand for capital through the solution of households and firms. The household problem is standard:

$$V(k, K) = \max_{c, k' > 0} \{U(c) + \beta V(k, K')\}$$

$$\text{s.t.} \quad c + k' = w(K) + (1 + r(K) - \delta)k \qquad \& \qquad K' = H(K)$$

- We solve using the FOC and the envelope, and find the Euler equation (household):

$$U'(\underbrace{w(K) + (1 + r(K) - \delta)k - k'}_{=c=g^c(k,K)}) = (1+r(K')-\delta)\beta U'(\underbrace{w(K') + (1 + r(K') - \delta)k' - k''}_{=c'=g^c(k',K')})$$

Production Externality: Firms

- The individual firm chooses its demand for k and l given prices and K:

$$\max_{k,l}\{k^{\alpha}l^{1-\alpha}K^{\gamma}-r(K)k-w(K)l\}$$

- FOC: $r(K)=\alpha k^{\alpha-1}l^{1-\alpha}K^{\gamma}$ & $w(K)=(1-\alpha)k^{\alpha}l^{-\alpha}K^{\gamma}$

- Using the FOC, we can find the individual firm's capital and labor demand:

$$k_j=\left(\frac{\alpha K^{\gamma}}{r(K)}\right)^{\frac{1}{1-\alpha}}l_j \qquad \text{and} \qquad l_j=\left(\frac{(1-\alpha)K^{\gamma}}{w(K)}\right)^{\frac{1}{\alpha}}k_j$$

Production Externality: Aggregate Demand

- We can find the aggregate labor demand by summing individual demands:

$$\int_0^1 l_j dj = \int_0^1 \left(\frac{(1-\alpha)K^\gamma}{w(K)}\right)^{\frac{1}{\alpha}} k_j dj = \left(\frac{(1-\alpha)K^\gamma}{w(K)}\right)^{\frac{1}{\alpha}} \underbrace{\int_0^1 k_j dj}_{=K} = L^d(K)$$

- The aggregate demand for capital is the sum of individual demands:

$$\int_0^1 k_j dj = \int_0^1 \left(\frac{\alpha K^\gamma}{r(K)}\right)^{\frac{1}{1-\alpha}} l_j dj = \left(\frac{\alpha K^\gamma}{r(K)}\right)^{\frac{1}{1-\alpha}} \underbrace{\int_0^1 l_j dj}_{=L(K)} = G^d(K)$$

Recursive Competitive Equilibrium: Externalities

- **Definition:** A recursive competitive equilibrium is a value function, V, decision rules (policy functions) g^k and g^c, demand functions (G^d, L^d), price functions (r, w), and aggregate movement law H, such that:

  1. Given functions r, w, and H, the value function V is the solution to the family's Bellman equation with decision rules g^k and g^c.
  2. Firms' demands satisfy:
  $$r(K) = \alpha G^d(K)^{\alpha-1} L^d(K)^{1-\alpha} K^\gamma \quad \text{and} \quad w(K) = (1-\alpha) G^d(K)^\alpha L^d(K)^{-\alpha} K^\gamma.$$
  3. Agents' expectations are rational (consistency):
  $$K' = H(K) = g^k(k, K) \quad \text{and} \quad C(K) = g^c(k, K)$$
  4. Markets for goods, capital, and labor are in equilibrium:
  $$C(K) + K' = G^d(K)^\alpha L^d(K)^{1-\alpha} K^\gamma + (1-\delta)K$$
  $$G^d(K) = K \quad \text{and} \quad L(K) = 1$$

Recursive Competitive Equilibrium: Externalities

- In the competitive equilibrium, firms do not consider the externality and demand less capital than the optimum, reducing $r(K)$.
- A lower interest rate incentivizes less savings by households. Note the Euler Equations:

$$\begin{aligned} \text{Planner} \quad & U'(C) = [1 + (\alpha+\gamma)(K')^{\alpha+\gamma-1} - \delta]\beta U'(C') \\ \text{Competitive Eq.} \quad & U'(C) = (1 + \underbrace{r(K')}_{\alpha(K')^{\alpha+\gamma-1}} - \delta)\beta U'(C') \end{aligned}$$

- ...and the steady-state capital:

$$K_{ss}^{PLAN} = \left[(\alpha+\gamma)/\left(\frac{1}{\beta} - (1-\delta)\right)\right]^{\frac{1}{1-\alpha-\gamma}} \quad \text{and} \quad K_{ss}^{RCE} = \left[\alpha/\left(\frac{1}{\beta} - (1-\delta)\right)\right]^{\frac{1}{1-\alpha-\gamma}}$$

- The planner internalizes the externality and accumulates more capital.

Markov Chains

Uncertainty in Dynamic Programming

- Up to the present moment, we have not specified the structure of uncertainty: in principle, an event may depend on the entire history of previous events.
- In macroeconomics, uncertainty will be essentially modelled as Markov Chains and first-order linear difference equations (e.g., an AR(1)).
- In other words, we will ignore the history and focus only on the last realization.
- But nothing prevents us from specifying more general stochastic processes!

Markov Chains

- **Definition**: Let $x_t \in X$, where $X = \overline{x}_1,\ \overline{x}_2, ..., \overline{x}_n$ is a finite set of values. A Stationary Markov Chain is a stochastic process $\{x_t\}_{t=0}^{\infty}$ defined by X, a transition matrix $P_{n\times n}$, and an initial probability distribution π_0 (a $1 \times n$ vector) for x_0.

- **Markovian Property**: A stochastic process $\{x\}$ possesses the Markovian Property if for every $k \geq 1$ and every t: $Prob(x_{t+1}|x_t, x_{t-1}, ..., x_{t-k}) = Prob(x_{t+1}|x_t)$.

- The elements of $P_{n\times n}$ represent the probabilities: $P_{ij} = Prob(x_{t+1} = \overline{x}_j|x_t = \overline{x}_i)$.

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots \\ \vdots & \ddots & \\ P_{n1} & & P_{nn} \end{bmatrix}$$

- For every i: $\sum_{j=1}^{n} P_{ij} = 1$.

Markov Chains

- The transition matrix defines the probabilities of moving from state i to state j in one period.

- The probability of transitioning from one state to another in two periods: P^2.

$$Prob(x_{t+2} = \overline{x}_j | x_t = \overline{x}_i) = \sum_{k=1}^{n} P_{ik} P_{kj} \equiv P_{ij}^{(2)},$$

  where $P_{ij}^{(2)}$ is the $(i,\ j)$ element of P^2.

- Given the vector π_0, π_1 is the unconditional probability of x_1: $\pi_1 = \pi_0 P$.

- Similarly: $\pi_2 = \pi_0 P^2$, $\pi_t = \pi_0 P^t$, and $\pi_{t+1} = \pi_t P$

Markov Chains

- **Definition**: An **invariant** unconditional distribution for P is a probability vector π such that $\pi = \pi P$.

- Thus, an invariant distribution satisfies:

$$\pi I = \pi P,$$
$$\pi I - \pi P = \pi[I - P] = 0.$$

  That is, π is an eigenvector of P (normalized $\sum_{i=1}^{n} \pi_i = 1$), with a unit eigenvalue.

- The fact that P has non-negative elements and rows that sum to one ensures that P has at least one eigenvector and eigenvalue.

- However, the invariant distribution is not necessarily unique.

Example: Unemployment-Employment

- Suppose a two states MC: first state represents the worker is employed, while the second unemployment.
- The probability that an employed worker separate is s, while the probability of an unemployed find a job is f.

$$P = \begin{bmatrix} 1-s & s \\ f & 1-f \end{bmatrix}$$

- Let the stationary distribution $\pi = [1-u_{ss}, u_{ss}]$, where u_{ss} is the SS unemployment.
- The u_{ss} satisfies the following equation:

$$u_{ss} = s(1-u_{ss}) + (1-f)u_{ss} \qquad \Leftrightarrow \qquad u_{ss} = \frac{s}{s+f}$$

Example: Stationary Distribution is Not Unique

- **Example**:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 0.5 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}$$

- The matrix P has two unit eigenvalues associated with the invariant distributions $\pi = [1\ 0\ 0]$ and $\pi = [0\ 0\ 1]$.

- Note that any initial distribution with zero mass on the second state is an invariant distribution.

- States 1 and 3 are absorbing states since once you enter them, you will never leave.

Markov Chains

- Let π_∞ be the unique vector that satisfies $\pi_\infty = \pi_\infty P$ and, for all initial distributions π_0:

$$\lim_{t \to \infty} \pi_0 P^t = \pi_\infty$$

- Then we can say that the Markov Chain is asymptotically stationary with a unique invariant distribution.
- **Theorem 1:** Let P be a transition matrix with $P_{ij} > 0 \; \forall (i,j)$. Then P has a unique invariant distribution, and the Markov Chain is asymptotically stationary.
- **Theorem: 2** Let P be a transition matrix with $P_{ij}^n > 0 \; \forall (i,j)$, for some $n \geq 1$. Then P has a unique invariant distribution, and the Markov Chain is asymptotically stationary.
- Intuitively, it must be possible to go from one state to another in one (Theorem 1) or n steps (Theorem 2).

Example: SIRD Model

- During COVID the SIRD model was pretty popular. It represents 4 states of a MC:
  - Susceptible, Infectious, Recovered, Death (in order).

$$P = \begin{matrix} S \\ I \\ R \\ D \end{matrix} \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Suppose all the population start at "Susceptible": $\pi_0 = [1\ 0\ 0\ 0]$.
- Eventually, everybody will be at the some convex combination of the last two states
- We call the last two states "absorbing states" and the first two "transient states".

Example: SIRD Model

- Suppose we have an additional (last) state called "zombie": with certain probability someone dead can come back as zombie. The zombies may die again.

$$P = \begin{matrix} S \\ I \\ R \\ D \\ Z \end{matrix} \begin{bmatrix} 0.9 & 0.1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0.1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

- Part of the (dead) population will be in a eternal loop in the last two states. This is called "ergodic set".

- The markov chain is said to be ergodic if all states are ergodic.

Stochastic Dynamic Programming

Stochastic Dynamic Programming

- The main advantage of dynamic programming appears when we introduce uncertainty.
- When we represent the stochastic process as a Markov Chain, only the last realization is sufficient $\Rightarrow$ there is no need to write the entire history.
- The theorems are similar to the previous ones with some modifications taking into account the stochastic process z.
  - References: Acemoglu or SLP.

Stochastic Dynamic Programming: Stochastic Growth

- Suppose z_t is a first-order Markov Chain with conditional density $f(z'|z)$.
  - The value of z contains information about the problem in period t and the expectation of period $t+1$.
- Suppose the production function depends on z as follows: zk^α.
- The Bellman equation for the neoclassical model:

$$V(k,z) = \max_{k' \in [0, zk^\alpha + (1-\delta)k]} u(zk^\alpha + (1-\delta)k - k') + \beta \mathbb{E}[V(k',z')|z],$$

where $\mathbb{E}[V(k',z')|z] = \int V(k',z') f(z'|z) dz'$.

- With the policy function: $k' = g(k,z)$.

Stochastic Dynamic Programming

- FOC:

$$u'(zk^{\alpha} + (1-\delta)k - k) = \beta \mathbb{E}[V_k(k', z')|z]$$

- And the envelope condition:

$$V_k(k', z') = u'(z'k'^{\alpha} + (1-\delta)k' - k'')(z'\alpha k'^{\alpha-1} + 1 - \delta)$$

- Stochastic Euler equation:

$$u'(c(k, z)) = \beta \mathbb{E}[u'(c(k', z'))(z'\alpha k'^{\alpha-1} + 1 - \delta)|z]$$

- Where $c(k, z)$ is the consumption policy function.

Stochastic Dynamic Programming

Growth Model (Stochastic)

- Shock: $z \in \mathbb{R}^+$.
- State: z and k.
- Control: k'.
- Feasible set: $k' \in \Gamma(k, z) = [0, zk^\alpha + (1-\delta)k]$.
- Return function: $F(k, z, k') = u(zk^\alpha + (1-\delta)k - k')$
- State law of motion: $(k', z') = h(k', z'; z, k) = (k', z')$ (trivial).
- Bellman's equation:

$$V(k, z) = \max_{k' \in \Gamma(k,z)} F(k, z, k') + \beta \mathbb{E}[V(k', z')|z]$$

Stochastic Dynamic Programming

General Form

- Shock: $z \in \mathbb{R}^n$.
- State: $x \in \mathbb{R}^l$ ($z^{(i)}$ may or may not be included!).
- Control: $y \in \mathbb{R}^m$.
- Feasible set: $y \in \Gamma(x),\ \Gamma : \mathbb{R}^l \to \mathbb{R}^m$.
- Return function: $F(x, y),\ F : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$.
- State law of motion: $x' = h(x, y, z'),\ h : \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^l$.
- Bellman's equation:

$$V(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta \mathbb{E}[V(\underbrace{h(x, y, z')}_{x'})|z]$$

Example: Consumption and Saving with Stochastic Income

- Suppose an individual (with infinite life) who consumes c, saves a, and has *iid* income w. Utility follows standard assumptions.
- Budget constraint:

$$c + a = w + (1 + r)a$$

- Bellman equation:

$$V(a, w) = \max_{c,a'} u(c) + \beta \mathbb{E}[V(a', w')|w]$$

- Policy functions: consumption $c(a, w)$ and saving $a'(a, w)$.

Example: Consumption and Saving with Stochastic Income

Cash-on-hand

- Define the variable "cash-on-hand": $x \equiv a(1+r)+w$.
  - State: x. Control: a' (or c).
  - $\Gamma(x)=[-b,x]$; $F(x,a')=u(x-a')$.
  - Law of motion: $x'=h(a',w')=a'(1+r)+w'$.
- The Bellman equation:

$$V(x)=\max_{a'\in[-b,x]} u(x-a')+\beta\mathbb{E}[V(h(a',w'))]$$

- State variables reduced from (a,w) to x.
- Often reducing the number of state variables turns out to be valuable for analytical and computational purposes (think about the curse of dimensionality).

Example: Consumption and Saving with Stochastic Income

- Solving the **cash-on-hand** problem. The FOC:

$$u'(x - a') = \beta \mathbb{E}[V_x(h(a', w'))h'(a', w')]$$

- Because the control is different than the state, we have to explicitly account for the law of motion of the state.
- Using the envelope condition $V_x = u'(x - a)$, we have the Euler equation:

$$u'(x - a')\beta(1 + r)\mathbb{E}[u'(x' - a'')]$$

Example: Consumption and Saving with Stochastic Income

- Euler equation:

$$u'(c) = \beta(1 + r)\mathbb{E}[u'(c')]$$

- What determines individuals' saving rate?

- Three motives:

  1. Intertemporal substitution: β vs $(1 + r)$.
  2. Consumption smoothing: desire to smooth differences in income across periods (e.g., retirement, expected increase in income, etc)
  3. Precautionary savings: insurance against future shocks.

- The first two does not need stochastic income.

Example: Precautionary Saving

- Suppose only two periods ($t = 0,\ 1$), $\beta(1+r) = 1$ and $a_0 = 0$. With only two periods we also have: $a_2 = 0$.
- First, imagine there is no randomness, and $w_1 = \overline{w}$. The (deterministic) Euler equation is:

$$u'(\underbrace{w_0 - a_1}_{c_0}) = u'(\underbrace{a_1(1+r) + \overline{w}}_{c_1}) \qquad \Rightarrow \qquad c_0 = c_1$$

  - Thus, the only savings motive is if w_0 is different than c_0.
- Now suppose w is stochastic (an increase in risk): $w = \overline{w} + \varepsilon$, where $\varepsilon \sim G(\sigma)$ with mean zero and variance σ. The (stochastic) Euler equation:

$$u'(c_0) = \mathbb{E}[u'(c_1)]$$

- **Problem:** the expected value is not necessarily equal $\mathbb{E}[u'(c_1)] \neq u'(c_1)$. We cannot say consumption will be the same in both periods!!!

Digression: Jensen's Inequality

- A function $f(x)$ is said to be convex if, for $\alpha \in [0,1]$ and two points x_1, x_2, it satisfies:

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

- Consider a random variable x that with probability p is x_1 and with $1-p$ is x_2.
  - In this case, the expected value of x is: $\mathbb{E}[x] = px_1 + (1-p)x_2$
  - Using the definition of convexity: $f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)]$

- For the general cases, this is called the Jensen's Inequality.
  - In our case, whether $\mathbb{E}[u'(c_1)] \lesseqgtr u'(c_1)$ depends on the convexity of the **marginal utility** $u'(c_2)$.

Example: Convexity of Marginal Utility

- In general, we assume utility is concave ($u''() < 0$), but what we are interested in is the second derivative of **marginal utility**.
  - $u''() < 0$ means marginal utility is decreasing.
  - $u'''() > 0$ means marginal utility is convex.

- If the third derivative of the utility is (strictly) increasing, marginal utility will be (strictly) convex and:

$$\mathbb{E}[u'(\underbrace{a_1(1+r)+\overline{w}+\varepsilon}_{c_1})] > u'(\mathbb{E}[a_1(1+r)+\overline{w}+\varepsilon]) = \underbrace{u'(a_1(1+r)+\overline{w}) = u'(c_1)}_{\text{Deterministic Mg. Utility in } t=1}$$

- Thus, convexity of mg. utility implies that $u'(c_0)$ is higher in the stochastic EE $\Rightarrow$ c_0 must be lower and savings, a_1, should be higher.

Example: Convexity of Marginal Utility

- The extra incentive to save given by uncertainty is called **precautionary saving**.
- Precautionary saving in this class of models requires $u'''() > 0$. This property is called **prudence**.
- Which utilities satisfy this property?
  - CRRA: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$.

$$u'(c) = c^{-\sigma} \qquad u''(c) = -\sigma c^{-(1+\sigma)} \qquad u'''(c) = \sigma(1+\sigma)c^{-(2+\sigma)} \qquad \checkmark$$

  - Quadratic utility: $u(c) = c - \theta\frac{c^2}{2}$.

$$u'(c) = 1 - \theta c \qquad u''(c) = -\theta \qquad u'''(c) = 0 \qquad \times$$

- Do not confuse **risk aversion** (concavity of u, or $u''(c) < 0$) with **prudence**.
  - You may still have precautionary savings without prudence if asset markets are incomplete!

Example: McCall Search Model

- Infinite time: $t = 1, 2, \ldots, \infty$.
- Each period, the agent receives an *iid* wage offer w from a c.d.f $F(w)$ with support $[0, \overline{\omega}]$.
- Decision:
  - If accept (A): receives w in the current period and forever (no dismissal).
  - If reject (R): receives $b \in (0, \overline{\omega})$ in the current period and receives a new offer in the next period.
- Linear utility and $\beta \in (0, 1)$: the agent maximizes: $\mathbb{E}_0[\sum_{t=0}^{\infty} y_t]$, where y_t is equal to b or w.
- Bellman equation:
  - Accept (A): $V^A(w) = \sum_{t=0}^{\infty} \beta^t w = \frac{w}{1-\beta}$,
  - Reject (R): $V^R(w) = b + \beta \mathbb{E}[V(w')] = b + \beta \int_0^{\overline{\omega}} V(w') f(w') dw'$.
  - $V(w) = \max\{V^A(w), V^R(w)\}$.

Example: McCall Search Model

- **Solution:** find the reservation wage w^*

$$g(w) = \begin{cases} \text{A if } w \geq w^*, \\ \text{R if } w < w^*. \end{cases} \tag{41}$$

- Characterized by the indifference condition:

$$V^A(w^*) = V^R(w^*) \longleftrightarrow \frac{w^*}{1-\beta} = b + \beta \int_0^{\overline{\omega}} V(w')f(w')dw' \tag{42}$$

- **Exercise:** show that

$$w^* - b = \frac{\beta}{1-\beta} \int_{w^*}^{\overline{\omega}} w' - w^* f(w')dw', \tag{43}$$

LHS: opportunity cost of rejecting the offer w^*. RHS: expected benefit of searching once more (*option value*).

- Show that w^* is increasing in b (hint: use the implicit function theorem).

Example: Firm with Exit Decision

- Suppose a firm with production function: $y = f(z, n) = zn^{\alpha}$ and $\alpha < 1$, where n is the number of hired workers, and z is productivity.
  - z follows a Markov chain with monotonically increasing conditional density $F(z'|z)$.
  - The firm hires workers in a competitive labor market for wage w, and sells the final good for $p = 1$.
- Every period, before knowing the realization of z, the firm decides whether to operate or exit the market.
  - If it decides to operate, it has to pay the fixed cost wc_f. If it decides to exit the market, it pays zero, but cannot operate again ever.
  - The firm discounts future profits by β.
- Write the firm's value function. What are the control(s), state(s), and return function? What can we say about the firm's exit decision and its productivity?

Example: Firm with Exit Decision

- Note that the per-period profit of a firm with productivity z is given by:

$$\pi(z; p, w) = \max_n \{pzn^\alpha - wn - wc_f\} \qquad \Rightarrow \qquad n^d(z; p, w) = \left(\frac{\alpha pz}{w}\right)^{\frac{1}{1-\alpha}}$$

  and the profit is:

$$\pi(z; p, w) = (1-\alpha)(pz)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} - c_f.$$

  - Suppose that for $c_f > 0$, s.t. there exists a z such that $\pi = 0$.

- Knowing z, the profit is given. The firm's only decision is to exit (or not) the market:

$$V(z) = \pi(z; p, w) + \beta \max\left\{ \int V(z')dF(z'|z), 0 \right\}$$

Example: Firm with Exit Decision

- Given that profit is increasing in z and F is monotonic, the value function is increasing in z.
- There exists a cut-off $\tilde{z}$ such that, for all $z < \tilde{z}$, the firm decides to exit.
- We can find the cut-off by equating the expected value of the firm with its exit value:

$$\mathbb{E}[V(z')|\tilde{z}] = \int V(z')dF(z'|\tilde{z}) = 0$$

- This does NOT mean that the firm will never have negative profit. It may have negative profit for some periods, if it expects a mean reversion of its z in the future.

Example: Value Function Iteration

- To solve the function on the computer, we will use the same method as before: iterate the value function with grid search for maximization.
- The only key change is how to deal with the Markov process.
- If it's discrete, we don't need to do anything!
- If it's continuous, we need to use some discretization method:
  - The most well-known ones (applied to an $AR(1)$): Tauchen and Rouwenhorst.
- There are alternative ways to compute a conditional expectation on the computer.
  - Remember that expectation is basically an integral $\rightarrow$ computing an expectation is computing an integral numerically.
- **Example:** Stochastic Growth Model (with $\delta = 1$).

Value Function Iteration

1. Discretize k into a vector with n_k points between $\underline{K}$ and $\overline{K}$. Define the points on the grid as $\{K_1, K_2, ..., K_I\}$.
2. Discretize z as a Markov process with n_z points. This implies a vector of values for $\{Z_1, Z_2, ..., Z_{n_z}\}$ and a transition matrix $P_{n_z \times n_z}$.
3. The value function will be stored in a matrix $n_k \times n_z$: $\{V_{ij}\}$. Initialize the matrix with your "guess' V^0 (each point of V_{ij} is the value associated with capital k_i and productivity z_j).
4. Compute the expectation of the value function ($\mathbb{E}[V_{ij}] = VP'$):

$$\underbrace{\begin{bmatrix} V_{11} & V_{12} & \dots & V_{1n_z} \\ \vdots & \ddots & \ddots & \\ V_{n_k1} & \dots & \dots & V_{n_kn_z} \end{bmatrix}}_{V_{n_k \times n_z}} \times \underbrace{\begin{bmatrix} P_{11} & P_{21} & \dots & P_{n_z1} \\ P_{12} & \ddots & \ddots & \\ P_{1n_z} & \dots & \dots & P_{n_zn_z} \end{bmatrix}}_{P'_{n_z \times n_z}} = \underbrace{\begin{bmatrix} \mathbb{E}[V_{11}] & \mathbb{E}[V_{12}] & \dots & \mathbb{E}[V_{1n_z}] \\ \vdots & \ddots & \ddots & \\ \mathbb{E}[V_{n_k1}] & \dots & \dots & \mathbb{E}[V_{n_kn_z}] \end{bmatrix}}_{\mathbb{E}[V]}$$

Note that $\mathbb{E}[V_{ij}] = \sum_m^{n_z} P_{jm} V_{im}$ (conditional expectation at j).

Value Function Iteration

- The rest is standard:
- Compute V_{ij}^{n+1} using the procedure for every i and j (*grid search* or *brute force*):

$$V_{ij,l}^{n+1} = \begin{cases} u(z_j f(k_i) - k_l) + \beta \mathbb{E} V_{lj}^{n}, & \text{if } z_j f(k_i) - k_l = c_{ij,l} > 0 \\ -\infty, & \text{if } z_j f(k_i) - k_l = c_{ij,l} \leq 0 \end{cases}$$
$$V_{ij}^{n+1} = \max\{V_{ij,1}^{n+1}, V_{ij,2}^{n+1}, ..., V_{ij,n_K}^{n+1}\}$$

- Calculate $d = \max_{i,\ j} |V_{ij}^{n+1} - V_{ij}^{n}|$. If $d < \varepsilon$, we have found the value function $V_{n+1} = V$. Otherwise, update the guess, $V_n = V_{n+1}$, and return to the previous point.

Continuous Time Dynamic Programming: The Hamilton-Jacobi-Bellman Equation

Dynamic Programming

- Just like in discrete time, we can represent the problem using Dynamic Programming.
- A more flexible approach, especially for introducing uncertainty, discrete choice, etc.
- Same solution but we have to solve a partial differential equation instead of an ordinary differential equation.
- Easier to bring the problem to the computer (we won't cover numerical methods).

Bellman's Principle

- How to find the value function in continuous time?
- Consider Bellman's principle of optimality to obtain the value function $V(t-\Delta t, a)$:

$$V(t-\Delta t, a) = \max_{c>0}\{u(c)\Delta t + e^{-\rho\Delta t}V(t, a')\}$$
$$s.t. \quad a' = a + (ra + w - c)\Delta t$$

  - $u(c)\Delta t$: utility flow between periods $t-\Delta t$ and t.
  - $e^{-\rho\Delta t}V(t, a')$: continuation value.
  - $(ra + w - c)\Delta t$: income and consumption flow.

- Rewrite the equation:

$$V(t-\Delta t, a) = \max_{c>0}\{u(c)\Delta t + e^{-\rho\Delta t}V(t, a + (ra + w - c)\Delta t)\}$$

Hamilton-Jacobi-Bellman Equation

- Define $g(\Delta t) \equiv e^{-\rho\Delta} V(t, a + [ra + w - c]\Delta t)$ and take the Taylor expansion around the point $\Delta t = 0$:

$$g(\Delta t) = g(0) + g'(0)\Delta t + o(\Delta)$$
$$g(\Delta t) \approx V(t,a) + \left(-\rho V(t,a) + V_a(t,a)\dot{a}\right)\Delta t$$

  where $V_a(t,a)$ is the derivative with respect to a.

- The value function:

$$V(t - \Delta t, a) = \max_{c>0} \left\{u(c)\Delta t + V(t,a) + \left(-\rho V(t,a) + V_a(t,a)\dot{a}\right)\Delta t\right\}$$

Hamilton-Jacobi-Bellman Equation

- Continuing:

$$V(t-\Delta t, a) = \max_{c>0} \{u(c)\Delta t + V(t,a) + (-\rho V(t,a) + V_a(t,a)\dot{a})\,\Delta t\}$$

$$\frac{V(t-\Delta t, a) - V(t,a)}{\Delta t} = \max_{c>0} \{u(c) - \rho V(t,a) + V_a(t,a)\dot{a}\}$$

- Taking the limit $\Delta t \to 0$ and we find the **Hamilton-Jacobi-Bellman Equation**:

$$-V_t(t,a) + \rho V(t,a) = \max_{c>0} \{u(c) + (ra + w - c)V_a(t,a)\}$$

Hamilton-Jacobi-Bellman Equation

Hamilton-Jacobi-Bellman Equation:

$$-V_t(t,a) + \rho V(t,a) = \max_{c>0} \{u(c) + (ra + w - c)V_a(t,a)\}$$

- Partial differential equation $(V_t(t,a))$.
- Assumption: V is differentiable in all its arguments.
- **Intuition:** $V_a(t,a)$ represents the marginal increase in value when wealth a increases marginally $\rightarrow$ note the connection with the multiplier μ!
- If the problem is stationary (variables constant over time): $V_t(t,a) = 0$ and value function does not depend on time.

Hamilton-Jacobi-Bellman Equation

- We are interested in solving the problem. Necessary condition (FOC w.r.t c):

$$u'(c^*(t,a)) = V_a(t,a)$$

- And the envelope condition. Differentiating the HJB w.r.t a (evaluated at the optimum $c^*(t,a)$):

$$-V_{ta}(t,a) + \rho V_a(t,a) = \frac{\partial c^*(t,a)}{\partial a} \underbrace{\left(u'(c^*(t,a)) - V_a(t,a)\right)}_{=0 \text{ (by FOC)}} + ...$$

$$...(ra + w - c^*(t,a))V_{aa}(t,a) + rV_a(t,a)$$

Hamilton-Jacobi-Bellman Equation

- Finally:

$$V_{ta}(t,a) + \dot{a}V_{aa}(t,a) = -(r-\rho)V_a(t,a)$$

- Define the optimal solution for $a^*(t)$ using the motion law and $c^*(t,a)$:

$$\dot{a}^*(t) = ra + w - c^*(t,a)$$

- Thus:

$$\frac{dV_a(t,a^*(t))}{dt} = V_{ta}(t,a^*(t)) + \dot{a}^*(t)V_{aa}(t,a^*(t))$$

- Note that $V_t(t,a)$ is the partial derivative with respect to the first argument and $dV(t,a)/dt$ is the total derivative (potentially the second argument depends on t).

Hamilton-Jacobi-Bellman Equation

- Using FOC, and the envelope condition evaluated at the optimum $a^*(t)$:

$$\begin{aligned}
\frac{\frac{d}{dt}V_a(t, a^*(t))}{V_a(t, a^*(t)} &= -(r - \rho) \\
\frac{\frac{d}{dt}u'(c_t)}{u'(c_t)} &= -(r - \rho) \\
\frac{u''(c_t)\dot{c}_t}{u'(c_t)} &= -(r - \rho)
\end{aligned}$$

- Finally we find the same **Euler Equation**!

A note about uncertainty in continuous time

Uncertainty

- Let's consider the simplest process in continuous time: Poisson processes (*jump processes*).
  - General cases require an introduction to stochastic calculus.
- Consider w_t following a Poisson process with states: $\{w_1, w_2\}$
- The transition rate between state i and j is given by: $\eta_{ij} \geq 0$.
- The conditional probability of "jumping"from state 1 to state 2 in the interval Δt

$$P(\text{jump to } w_2 \text{ in interval } [t, t+\Delta t] | w_t = w_1) = 1 - e^{-\eta_{12}\Delta t} \approx \eta_{12}\Delta t + o(\Delta t)$$

where the approximation is valid for a Δt close to zero.

Uncertainty

- For N states $w \in \{w_1, ..., w_N\}$ with small Δt:

$$P(\text{jump to } w_j \text{ in interval } [t, t+\Delta t] | w_t = w_i) \approx \eta_{ij}\Delta t + o(\Delta t)$$

$$P(\text{stay in } w_1 \text{ in interval } [t, t+\Delta t] | w_t = w_i) \approx \left(1 - \sum_{j \neq i} \eta_{ij}\Delta t + o(\Delta t)\right)$$

- Probability of 2 jumps in the same interval is second order and disappears quickly as $\Delta t \to 0$

$$P(2 \text{ or + jumps: } i \to k \to j \text{ in } [t, t+\Delta t] | w_t = w_i) \approx \eta_{ji}\Delta t \times \eta_{kj}\Delta t = \eta_{ji}\eta_{ji}(\Delta t)^2$$

Consumption and Savings

- Consider the consumption and savings problem with $w \in \{w_1, w_2\}$ and $\eta_{12} = \eta_{21} = \eta$:

$$V(t-\Delta t, a, w_1) = \max_{c>0} \{u(c)\Delta t + e^{-\rho \Delta t}\left[(1-\eta\Delta t)V(t,a',w_1) + \eta \Delta t V(t,a',w_2)\right]\}$$

  with $a' = a + (ra + w_1 - c)\Delta t$.

- $(1-\eta\Delta t)V(t,a',w_1)$ continuation value when w does not change
- $\eta\Delta t V(t,a',w_2)$ continuation value when $w_1 \rightarrow w_2$.

Consumption and Savings

- Differentiate with respect to Δt and evaluate $\Delta t = 0$ (the result will be the same if we use the Taylor series as before):

$$\frac{\partial V(t-\Delta t, a, w_1)}{\partial \Delta t}\Big|_{\Delta t=0} = -V_t(t,a,w_1) =$$
$$= \max_{c>0}\{u(c) + \underbrace{\dot{a}V_a(t,a,w_1)}_{\text{Additional Savings}}\} - \rho V(t,a,w_1) + \underbrace{\eta(V(t,a,w_2) - V(t,a,w_1))}_{\text{Wage Differential}}$$

  Note that the effect of saving in state w_2 is second order.

- The solution satisfies the two symmetric HJBs:

$$-V_t(t,a,w_1) + (\rho+\eta)V(t,a,w_1) = \max_{c>0}\{u(c) - [ar + w_1 - c]V_a(t,a,w_1)\} + \eta V(t,a,w_2)$$
$$-V_t(t,a,w_2) + (\rho+\eta)V(t,a,w_2) = \max_{c>0}\{u(c) - [ar + w_2 - c]V_a(t,a,w_2)\} + \eta V(t,a,w_1)$$

Consumption and Savings

- To find the Euler Equation we use the same idea as before. Suppose the problem is stationary. The FOC of c implies:

$$u'(c^*(a,w)) = V_a$$

- The envelope condition (∂HJB$/\partial a$):

$$\rho V_a(a,w) = \eta(V_a(a,\tilde{w}) - V_a(a,w)) + rV(a,w) + \dot{a}V_{aa}(a,w)$$

- Ok, in discrete time we know that our EE is about the expected value of the marginal utility of $t+1$ (i.e. $\mathbb{E}[u'(c_{t+1})]$).
- What is the appropriate definition of this expected value in continuous time?

Consumption and Savings

- **Definition.** For the differentiable function f, define the *Infinitesimal Generator* as the operator $\mathcal{A}$:

$$\mathcal{A}f(a,w) = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t[f(a_{t+\Delta t}, w_{t+\Delta t})] - f(a_t, w_t)}{\Delta t}$$

- **Intuition**: The *Infinitesimal Generator* describes how the stochastic process evolves over a time interval.

- Our stochastic process depends on two variables: a_t^* and w_t. Basically $\mathcal{A}$ tells us how the function f evolves in expected value given a_t^* and w_t for an instant t.

Consumption and Savings

- In our case:

$$\mathbb{E}_t[f(a^*_{t+\Delta t}, w_{t+\Delta t})] \approx (1 - \eta \Delta t) f(a^*_t + \dot{a}, w_t) + \eta \Delta t f(a^*_t + \dot{a}, \tilde{w}_t)$$

- It is relatively easy to use the definition and demonstrate that:

$$\mathcal{A} f(a^*, w) = \underbrace{\dot{a} f_a(a, w)}_{\textit{Drift in } a} + \underbrace{\eta (f(a^*, \tilde{w}) - f(a^*, w))}_{\text{Wage transition}}.$$

- The envelope condition can be written:

$$\underbrace{\dot{a} V_{aa}(a, w) + \eta (V_a(a, \tilde{w}) - V_a(a, w))}_{\mathcal{A} V_a(a,w)} = -(r - \rho) V_a(a, w)$$

Consumption and Savings

- Combining the FOC with the envelope:

$$\frac{\mathcal{A}V_a(a^*, w)}{V_a(a^*, w)} = -(r - \rho)$$
$$\frac{\mathcal{A}u'(c^*(a, w))}{u'(c^*(a, w))} = -(r - \rho)$$

- Finally we find the Euler Equation!
- Where $\dfrac{\mathcal{A}u'(c^*)}{u'(c^*)}$ is the expected growth rate of u'.