Macroeconomics I

Foundations of Dynamic General Equilibrium Models: Equilibrium, Welfare, & Uncertainty

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Introduction

- We have see that the Solow model can match some stylized facts about economic growth through the production function.
- Nevertheless, the model lies in a strong assumption: an exogenous constant savings rate.
- We want to explicitly microfound the savings decision so it responds to changes in economic policy.
- Moreover, we want our model to be set in general equilibrium so prices respond to changes in the environment.
- In this lecture we will build these foundations so we can include them in our models in the future.

What We Learn in This Chapter

- How to write a dynamic macro model.
- How to define a competitive equilibrium.
- How to solve for the equilibrium prices and allocations.
- What can we say about welfare (in the Pareto sense).
- How to include uncertainty in a dynamic model.

References

- Dirk Krueger Ch. 2 and 6.
- PhD Macrobook Ch. 5, 6 and 7.
- Acemoglu Ch. 5.
- Ljungqvist and Sargent. Ch. 8.

A (Macro)economic Model

Building a (Macro)economic Model

- **Preferences:** Utility function.
- **Technology:** Production function.
- **Government:** Policy instrument, objective function.
- **Environment:** Information, market structure, goods, population, etc.
- **Endowments:** Agents' endowments.
- **Concept of equilibrium:** How prices are defined, or alternatively, how interactions between the agents occur.

We can define the prices and allocations of the economy in with this information.

Competitive Equilibrium

We will focus on a competitive equilibrium.

Definition (Competitive Equilibrium)

A competitive equilibrium consists of allocations (a list/vector of quantities) and prices (list/vector of prices) such that:

(i) Given the prices, the allocations solve the agents' problem.

(ii) The allocations respect the economy's resource constraints (i.e., they are feasible allocations).

- A set of equations describing the actions of agents and the constraints of the economy in a way that prices describe an equilibrium (no excess demand or supply).
- The second condition implies that all markets are in equilibrium (i.e., market clearing).

Solving the Model

Step by step:

1. Describe the *"environment"*.
2. Solve the individual problem of each agent.
   - Write the maximization problem and the set of equations that determine the solution.
   - Household consumption (as a function of income and price), $c = f(y, p)$; firm's demand for labor (as a function of wage), $n = h(w)$, etc.
3. Specify the equilibrium conditions (*market clearing conditions*).
   - The aggregate demand for bananas must be equal to the aggregate supply of bananas, and the same for apples, etc.
4. Describe the competitive equilibrium.
   - Write all endogenous objects (prices, allocations, etc.) and all equations (agents' first-order conditions, market clearing, etc.), and eventually government policies.
   - System of N equations and N endogenous objects.

Solving the Model

Advantages of this approach

- Aggregate relationships respect individual constraints.
- **Transparency:** Clear map of what is preference/technology and what is the agents' endogenous decision.
- Agents' expectations are consistent with the model.
- Micro $\Rightarrow$ Macro.
- Policy changes impact the welfare of each individual agent.
- Testable implications about individual behavior.

(Macro)economic Models

Macro models are...

D ynamic

S tochastic

G eneral

E equilibrium

Two-Agent Endowment Economy

- **Environment:** 2 consumers ($i = 1, 2$) of a single good living infinitely many periods.
- **Preferences:**

$$U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t^i) \quad i = 1, 2. \tag{1}$$

- **Endowments:** Deterministic sequences $\{e^i\}_{t=0}^{\infty}$, where:

$$e_t^1 = \begin{cases} \hat{e}, & \text{if } t \text{ is even.} \\ 0, & \text{if } t \text{ is odd.} \end{cases} \quad e_t^2 = \begin{cases} 0, & \text{if } t \text{ is even.} \\ \hat{e}, & \text{if } t \text{ is odd.} \end{cases} \tag{2}$$

  and $\hat{e} > 0$.
- **"Technology":** The endowment can be transformed into a final consumption good at no cost: $c_t = \hat{e}$

Digression: A Note on Infinite Horizon

In macroeconomics, it is common to solve dynamic problems of infinite sum. Intuition behind $T = \infty$?

(i) **Altruism:** We derive utility from the well-being of our descendants. An agent who lives for one period and discounts the utility of their children with β:

$$U(c_\tau) = u(c_\tau) + \beta U(c_{\tau+1}) = \sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_t) \tag{3}$$

(ii) **Simplification:** When T is sufficiently high, the behavior of the model is similar to $T = \infty$. Models with $T = \infty$ are stationary and easier to work with.

Digression: A Note on Infinite Horizon

- When dealing with an infinite horizon, we need to ensure that:

$$U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4}$$

  is bounded.
- How to compare two consumption sequences $\{c_t\}_{t=0}^{\infty}$ that yield infinite U?
- Depending on the problem, this imposes restrictions on parameters and functional forms.
- If $c_t = \bar{c}$ is constant, the condition for the series to converge is $\beta < 1$.
- But if the sequence is of the form $\{c_t\}_{t=0}^{\infty} = \{c_0(1+\gamma)^t\}_{t=0}^{\infty}$, it will depend on γ, β, and $u(.)$.

Digression: Utility Function

- Assuming an exponential discount factor $\beta \in (0, 1)$, the utility function in its general form is given by

$$U^i(c_1^i, c_2^i, ..., c_T^i) \equiv \sum_{t=0}^{T} (\beta^i)^t u^i(c_t^i), \tag{5}$$

  where U is the utility function defined over a consumption sequence $\{c_t\}_{t=0}^{T}$.

- Exponential discounting implies that regardless of the period t, the discount between t and $t+1$ is always the same.

- T can be finite or infinite.

- The utility can be individual i-specific (but the problem becomes harder to solve).

Digression: Utility Function

We assume that $u()$:

- is a twice-differentiable function, strictly increasing $(u'(c) > 0)$, strictly concave $(u''(c) < 0)$, does not change over time, and does not depend on the decisions of other individuals.
- is *time-separable*.
- defined over $c > 0$.
- And that the marginal utility satisfies:

$$\lim_{c \to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \to \infty} u'(c) = 0 \tag{6}$$

  - This ensures that the agent's choice is always $c \in (0, \infty)$.
  - More consumption is always better, but an additional unit of c increases $\Rightarrow$ Marginal utility is decreasing.

Usual Utility Functions

- Usual utility functions used in macro models:

$$
\begin{aligned}
u(c) &= \ln c && \text{Log} \\
u(c) &= \frac{c^{1-\sigma}-1}{1-\sigma}, \quad \sigma > 0 && \text{CRRA} \\
u(c) &= \theta c, \quad \theta > 0 && \text{Linear} \\
u(c) &= c - \theta \frac{c^2}{2} \quad \theta > 0 && \text{Quadratic} \\
u(c) &= -\frac{\exp\{-\alpha c\}}{\alpha} \quad \alpha > 0 && \text{CARA}
\end{aligned}
$$

- Note that $\ln c$ is a special case of CRRA when $\sigma = 1$.

Market Structure

- What is a decentralized equilibrium? Allocations supported by prices that clear all markets.
- Basically solving supply and demand in $N-1$ markets (by Walras' law, the N-th market will be in equilibrium).

Two ways to represent a competitive equilibrium in a dynamic economy

1. **Arrow-Debreu:** All exchanges occur in period 0.
2. **Sequential Markets:** Markets open each period.

Arrow-Debreu

Arrow-Debreu Structure

- Agents "trade" in period 0 (or sign a contract with perfect commitment).
- In subsequent periods, they only deliver the quantities agreed upon in period 0.
- The price of the final consumption good is p_t in each t. We normalize $p_0 = 1$.
- Intuitively, a consumption good at t is a different commodity at $t-1$ (thus has a different price).
- In *complete markets*, T periods are equivalent to having T different goods in a single period.
- The budget constraint for agent i in period 0: $\sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t e_t^i$.

Arrow-Debreu

Definition. A competitive Arrow-Debreu equilibrium is a sequence of allocations $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ and prices $\{p_t\}_{t=0}^{\infty}$ such that:

1. Given the price sequence $\{p_t\}_{t=0}^{\infty}$, for $i = 1, 2$, $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ is the solution to the problem:

$$\max_{\{c_t^i \geq 0\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{7}$$

$$s.t. \quad \sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t e_t^i \tag{8}$$

2. The good**s** market is in equilibrium:

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 = \hat{e} \quad \forall t \tag{9}$$

We have described the environment of the economy and the definition of competitive equilibrium; let's go to the optimization problem of agents $i = 1, 2$.

Solving the Two-agents Problem

- Suppose $u(c) = \log(c)$ $\beta \in (0, 1)$. For an arbitrary agent $i = 1, 2$:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log\left(c_t^i\right) + \lambda_i \left(\sum_{t=0}^{\infty} p_t e_t^i - \sum_{t=0}^{\infty} p_t c_t^i \right) \tag{10}$$

  where λ_i is the Lagrange multiplier for the budget constraint of agent i.

  - The solution is interior: $c_t > 0$ for every t ($\lim_{c \to 0} u'(c) = \infty$).
  - The budget constraint holds with equality (u is strictly increasing).

- FOC: $\dfrac{\beta^t}{c_t^i} = \lambda_i p_t$ for $t = 0, 1, .., \infty$.

- Solving for λ_i in two arbitrary periods:

$$\frac{1}{c_t^i} = \frac{p_t}{p_{t+1}} \frac{\beta}{c_{t+1}^i} \quad \text{for all } t \text{ and } i = 1, 2 \tag{11}$$

Solving the Two-agents Problem

- Okay, a system of infinite equations, now what? Note that: $c_t^i = c_0^i \frac{p_0}{p_t} \beta^t$.
- Substituting into the budget constraint and normalizing $p_0 = 1$:

$$\sum_{t=0}^{\infty} p_t e_t^i = \sum_{t=0}^{\infty} p_t c_t^i = c_0 \sum_{t=0}^{\infty} \beta^t = \frac{c_0^i}{1-\beta} \tag{12}$$

- This gives us the sequence of allocations as a function of prices.
- To complete the solution, we need to find the prices that support the equilibrium.

Solving the Two-agents Problem

- Equilibrium in the goods market:

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 = \hat{e} \quad \forall t \tag{13}$$

- Summing the FOCs of both agents:

$$c_{t+1}^1 + c_{t+1}^2 = \beta \frac{p_t}{p_{t+1}} (c_t^1 + c_t^2) \quad \forall t \tag{14}$$

- This implies $\hat{e} = \beta \frac{p_t}{p_{t+1}} \hat{e} \Leftrightarrow \beta = \frac{p_{t+1}}{p_t}$. With the normalization $p_0 = 1$:

$$p_t = \beta^t \quad \forall t \tag{15}$$

- Meaning that $c_{t+1}^i = c_t^i = c_0^i$ for both i.

Solving a Dynamic Problem

- We have the equilibrium solution, but we can go further and show the consumption sequence as a function of parameters.
- Agent 1 receives the endowment first, thus:

$$\sum_{t=0}^{\infty} p_t e_t^1 = \hat{e} \sum_{t=0}^{\infty} \beta^{2t} = \frac{\hat{e}}{1-\beta^2} \tag{16}$$

- Similarly, we can show that for agent 2:

$$\sum_{t=0}^{\infty} p_t e_t^2 = \frac{\hat{e}\beta}{1-\beta^2} \tag{17}$$

- Finally, the equilibrium allocations are given by:

$$c_t^1 = c^1 = \frac{\hat{e}}{1+\beta} > \frac{\hat{e}}{2} \quad \text{and} \quad c_t^2 = c^2 = \frac{\hat{e}\beta}{1+\beta} < \frac{\hat{e}}{2} \tag{18}$$

- Agent 1 consumes more because she receives the endowment first.

Sequential Market Structure

- Agents "trade" every period and can borrow or lend at a one-period interest rate r_t.
- Define a_t as the agent's net position, i.e., savings from period $t-1$.
- The price of the final consumption good is p_t in each t. We normalize $p_t = 1$ *in all periods*.
- The budget constraint for agent i in period t:

$$c_t + a_{t+1} \leq a_t(1 + r_t) + e_t^i. \tag{19}$$

- Alternatively, we can use the price of a one-period bond as $q_t \equiv 1/(1 + r_t)$.

Sequential Markets

Definition. A Sequential Markets equilibrium is a sequence of allocations $\{c_t^1, c_t^2, a_{t+1}^1, a_{t+1}^2\}_{t=0}^{\infty}$ and prices $\{r_t\}_{t=0}^{\infty}$ given that:

1. For $i = 1, 2$, given the sequence of interest rates $\{r_t\}_{t=0}^{\infty}$, $\{c_t^1, c_t^2, a_{t+1}^1, a_{t+1}^2\}_{t=0}^{\infty}$ is the solution to the problem:

$$\max_{\{c_t^i > 0, a_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{20}$$

$$s.t. \quad c_t + a_{t+1} \leq a_t(1 + r_t) + e_t^i \quad \forall t,\ a_0^i = 0 \tag{21}$$

$$\lim_{T \to \infty} \frac{a_{T+1}}{\Pi_{t=0}^{T}(1 + r_t)} \geq 0 \quad \textit{(No-Ponzi-game)} \tag{22}$$

2. The goods and assets (bonds) markets are in equilibrium:

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 = \hat{e} \quad \forall t \tag{23}$$

$$a_{t+1}^1 + a_{t+1}^2 = 0 \quad \forall t \tag{24}$$

Sequential Markets

- Note that there is an additional equilibrium condition in the asset market, and there are infinite restrictions.
- If markets are **complete** and the ***no-Ponzi game*** restriction is satisfied, an Arrow-Debreu equilibrium always has an equivalent in Sequential Markets.
  - See the theorem and proof in DK's notes.
- What does the ***no-Ponzi game*** restriction mean?
  - To give intuition, let's solve the sequential problem in **finite time** and replace NPG with a restriction $a_{T+1} \geq 0$.

Sequential Markets in Finite Horizon

Kuhn-Tucker for agent i:

$$\mathcal{L} = \sum_{t=0}^{T} \left[\beta^t u(c_t^i) + \lambda_t \left(e_t^i + a_t^i(1+r_t) - c_t^i - a_{t+1}^i\right)\right] + \mu_T a_{T+1}^i \tag{25}$$

- Kuhn-Tucker conditions:
  - $a_{T+1} \geq 0$, $\lambda_t \geq 0$, and $\mu_t \geq 0$.
  - Complementary slackness: $a_{T+1}\mu_T = 0$

First-order conditions...

$$u'(c_t)\beta^t = \lambda_t \quad \text{and} \quad \lambda_t = (1+r_{t+1})\lambda_{t+1} \quad \text{for} \quad t = 0, ..., T-1$$
$$u'(c_T)\beta^T = \lambda_T \quad \text{and} \quad \lambda_T = \mu_T \quad \text{for} \quad t = T$$

Sequential Markets in Finite Horizon

- **Using the FOCs**:

$$u'(c_t) = (1 + r_{t+1})\beta u'(c_{t+1}) \quad t = 0, 1, ..., T - 1$$

- This is the ***Euler Equation*** $\Rightarrow$ the most important equation in modern macroeconomics.
  - Describes the trade-off between consumption and savings for the household.
- **In period T**:

$$\beta^T u'(c_T) = \lambda_T = \mu_T > 0$$

  - Since $c_T > 0$ and $u'(c_T) > 0$ implies $\mu_T > 0$.
  - Due to the complementary slackness in the KT conditions, $a_T = 0!$ $\Rightarrow$ the agent doesn't want to die with "money in the pocket".
  - What would happen if we didn't have the restriction $a_T \geq 0$? What does this tell us about the No-Ponzi game?

No-Ponzi Game

- *No-Ponzi game* condition: without it, the agent could always roll over the debt and achieve a higher consumption sequence.
- Substituting the budget constraints up to T (assuming equality):

$$\frac{c_0 - e_0}{(1+r_0)} + \frac{a_1}{(1+r_0)} = a_0$$

$$\frac{c_1 - e_1}{(1+r_1)} + \frac{a_2}{(1+r_1)} = a_1 ...$$

$$\Rightarrow \sum_{t=0}^{T} \frac{c_t - e_t}{\Pi_{j=0}^{t}(1+r_j)} + \underbrace{\frac{a_{T+1}}{\Pi_{t=0}^{T}(1+r_t)}}_{=0 \text{ No-Ponzi-game}} = a_0$$

- Alternatively to this condition, we can impose a lower bound such that:

$$a_{t+1} \geq -\overline{A}, \tag{26}$$

provided this lower bound is high enough not to restrict the choice of a_{t+1}.

Sequential Markets in Infinite Horizon

- In infinite time, we don't have the final condition. Using the **Euler Equation** (assuming log):

$$\frac{1}{c_t^i} = (1 + r_{t+1})\beta \frac{1}{c_{t+1}^i} \quad \forall t \text{ and } i = 1, 2.$$

- Note that:

$$c_1^i = (1 + r_1)\beta c_0^i \quad \& \quad c_2^i = (1 + r_2)\beta c_1^i \quad \rightarrow \quad c_2^i = (1 + r_2)(1 + r_1)\beta^2 c_0^i$$

$$\Rightarrow \qquad c_t^i = c_0^i \beta^t \left[\Pi_{j=1}^t (1 + r_j)\right]$$

$$c_t^i = c_0^i \frac{\beta^t}{1 + r_0} \left[\Pi_{j=0}^t (1 + r_j)\right]$$

Sequential Markets in Infinite Horizon

- Substituting $c_t^i = c_0^i \frac{\beta^t}{1+r_0}\left[\Pi_{j=0}^t (1+r_j)\right]$ into the intertemporal budget constraint:

$$\sum_{t=0}^{\infty} \frac{c_t^i}{\Pi_{j=0}^t (1+r_j)} + \lim_{T\to\infty} \frac{a_{T+1}}{\Pi_{t=0}^T (1+r_t)} = a_0^i + \sum_{t=0}^{\infty} \frac{e_t^i}{\Pi_{j=0}^t (1+r_j)}$$

$$\sum_{t=0}^{\infty} c_0^i \frac{\beta^t}{(1+r_0)} = a_0^i + \sum_{t=0}^{\infty} \frac{e_t}{\Pi_{j=0}^t (1+r_j)}$$

$$\frac{c_0^i}{(1-\beta)} = a_0^i (1+r_0) + \sum_{t=0}^{\infty} \frac{e_t}{\Pi_{j=1}^t (1+r_j)}$$

- Assuming $a_0^i = 0$ (could be positive or negative, wouldn't make a difference).
- Without NPG, the intertemporal budget constraint is not bound.

Prices

- Summing the Euler equation of the two agents: $c_t^1 + c_t^2 = (1 + r_{t+1})\beta(c_{t+1}^1 + c_{t+1}^2)$
- Using the goods market equilibrium equation: $c_t^1 + c_t^2 = e_t^1 + e_t^2 = \hat{e}$ for all t:

$$\hat{e} = (1 + r_{t+1})\beta\hat{e} \quad \Rightarrow \quad 1 + r_t = \frac{1}{\beta} \quad \forall t > 0.$$

  - Note that $1/(1 + r_{t+1}) = p_{t+1}/p_t$ from the Arrow-Debreu structure.
- Additionally, $c_t^i = c_0^i \quad \forall t$.

Solution

- Finally, using $\beta = 1/(1+r_t)$:

$$\frac{c_0^i}{(1-\beta)} = \sum_{t=0}^{\infty} \frac{e_t}{\Pi_{j=1}^t (1+r_j)} = \sum_{t=0}^{\infty} \beta^t e_t^i$$

  substituting the endowment sequences e_t^i of each agent i, we find the same allocations as the Arrow-Debreu market.

- Once we have the consumption of each agent in each period c_t^i, we can use the budget constraints and calculate their savings!

- Remember that the equilibrium in the asset market is: $a_t^1 + a_t^2 = 0$ for all t.

The Social Planner and the Welfare Theorems

The Social Planner

- In the previous section, we solved the model (i.e., the eq. allocations and prices) by finding the *decentralized equilibrium*:
  - **Decentralized equilibrium:** Find the price vector that supports the optimal allocations and CLEARS ALL MARKETS.
- We can also solve for the *optimal* allocations by solving the **social planner's problem**.
- **The "Benevolent" Social Planner's Problem**:
  - Maximize the utility of the HHs subject to the technological restrictions and resource constraints (NOT BUDGET CONSTRAINTS).
  - Does NOT involve any prices.
  - The solution(s) are the Pareto optimal allocations.

Welfare and Equilibrium

- Okay, we've found the solution for the **social planner**. What now?

- Close relationship between solving the planner's problem and the decentralized competitive equilibrium.

- Under certain conditions, the two problems result in the same allocations $\Rightarrow$ Welfare Theorems.
    - **First Welfare Theorem:** Competitive Equilibrium $\Rightarrow$ Pareto Optimal Allocations.
    - **Second Welfare Theorem:** Pareto Optimal Allocations $\Rightarrow$ Competitive Equilibrium.

- In this case, we can also say that the economy is Pareto efficient.

Pareto Optimality

- Suppose an arbitrary economy:
  1. N goods indexed by j;
  2. H families indexed by h consuming x_j^h with utility U^h and endowments e^h;
  3. F firms indexed by f producing y_j^f.

  The firm's ownership fraction is given by θ_h^f, where $\sum_h^H \theta_h^f = 1$.
- **Definition:** An allocation $\{x_j^h, y_j^f\}_{f\in F,\ h\in H,\ j\in N}$ is "feasible"if for every $j \in N$:

$$\sum_h^H x_j^h \leq \sum_h^H e_j^h + \sum_f^F y_j^f \tag{27}$$

- **Definition:** An allocation $\{x_j^h, y_j^f\}_{f\in F,\ h\in H,\ j\in N}$ is Pareto optimal if:
  1. it is "feasible";
  2. there is no other "feasible"allocation $\{\hat{x}_j^h, \hat{y}^f\}$ such that

$$U^h(\{\hat{x}_j^h\}_{j\in N}) \geq U^h(\{x_j^h\}_{j\in N}) \quad \text{for every } h \tag{28}$$

$$U^h(\{\hat{x}_j^h\}_{j\in N}) > U^h(\{x_j^h\}_{j\in N}) \quad \text{for at least one } \ h. \tag{29}$$

First Welfare Theorem

Theorem (First Welfare Theorem)

Suppose that $\{x_j^h, y_j^f, p_j\}$ is a competitive equilibrium, and all U^h are locally nonsatiated. Then $\{x_j^h, y_j^f\}$ is Pareto optimal.

- **Proof:** By contradiction. Suppose $\{x_j^h, y_j^f\}$ is not Pareto optimal (i.e., there exists another feasible allocation that gives more utility to at least one h) and use the definition of a competitive equilibrium.
- Note that we are assuming the existence of a competitive equilibrium (which may not exist depending on the form of U^h, and the sets of x and y).
- Pareto optimality says nothing about equity (an individual consuming everything is efficient).
- **When does the First Welfare Theorem not apply?**
  - Externalities; Incomplete Markets; Imperfect Competition; Asymmetric Information; Distortionary Taxation;

Second Welfare Theorem

Theorem (Second Welfare Theorem)

Consider the Pareto optimal allocation $\{x_j^h, y_j^f\}$*. Given certain conditions (convex production and consumption set, utility is concave, continuous, and locally nonsatiated), there exists a competitive equilibrium with prices* $\{p_j\}$ *and endowments* $\{e^h, \theta_h^f\}$ *that supports the allocation* $\{x_j^h, y_j^f\}$*.*

- **Proof:** The proof is more complicated as it implicitly involves demonstrating the existence of a competitive equilibrium. Basically, it involves showing the existence of prices (on a hyperplane) that support the allocations.
- Intuitively, the Second Welfare Theorem tells us that an allocation is part of a competitive equilibrium.
- Given an appropriate redistribution of initial endowments, we can pick the Pareto optimal allocation that is a competitive equilibrium.

Social Planner

- The Welfare Theorems say that we can go from a Pareto optimal allocation to a decentralized equilibrium and vice versa.
- Under certain conditions, it is sufficient to compute the Pareto optimal allocations by solving the problem of the *Social Planner* (which is generally simpler).
- **Negishi's Method**: Selects the appropriate weight according to the initial endowments of each family to find the allocations of the competitive equilibrium!

Planner's Problem

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [\alpha u(c_t^1) + (1-\alpha) u(c_t^2)]$$

$$s.t. \quad c_t^1 + c_t^2 = e_t^1 + e_t^2 = \hat{e}_t \quad \text{for all } t$$

$$c_t^i \geq 0 \quad \text{for all } t \text{ and for all } i$$

- The $\alpha \in [0, 1]$ defines the relative Pareto weights (e.g., if $\alpha = 0.5$ the social planner gives equal weight to the agents).
- The set of Pareto efficient consumption is a function of α: $c_t^i(\alpha)$.
- There is a α where $c_t^i(\alpha)$ coincides with the decentralized equilibrium (Negishi's method).
- **Exercise:** Solve the two-agent problem using Negishi's method.

Equilibrium in the Two-agent Problem

Summary:

- **The decentralized equilibrium**: sequential markets structure vs Arrow-Debreu $\Rightarrow$ "Market" equilibrium.
  - If markets are complete, the solutions are identical.
- **The benevolent planner's problem**: Gives the pareto optimal allocations.
- If the welfare theorems are satisfied, the two solutions are identical, and the equilibrium is optimal.

Uncertainty in General Equilibrium

Notation

- An event in period t: $s_t \in S$. S is the set of all possible events, which we assume is finite and equal for all t.
- An event history is a vector represented by: $s^t = (s_0, s_1, ..., s_t)$.
- Formally $s^t \in S^t$, where $S^t = S \times S \times S ... \times S$.
- The probability of observing a particular history of events is given by: $\pi(s^t)$.
- The conditional probability of observing s^t after the realization of s^τ: $\pi(s^t | s^\tau)$.
- In some places, you may also find the representation of a sub-history of s^t as: $s^t_{\rightarrow t-1}$.

Notation

- The goods in the economy, instead of being “just” indexed by t, also have to be indexed by the history of events s^t: $c_t(s^t)$.
- An agent chooses a consumption sequence dependent on the history of events: $\{c_t(s^t)\}_{t=0}^{\infty}$.
- Agents maximize the **expected utility**:

$$U(\{c_t(s^t)\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t \in S^t} \pi(s^t) u(c_t(s^t)) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]. \tag{30}$$

- **Example:** Two-agent endowment economy.
- Agents $i = \{1, 2\}$ receive an endowment $e_t^i(s^t)$ depending on the history s^t.

Market Structure: Arrow-Debreu

- Trades occur in period 0 before any uncertainty is realized.
- In period 0, agents trade consumption claims in all periods and *possible realizations of* s^t.
- Define the price of a unit of a consumption claim at t and s^t: $p_t(s^t)$.
- The budget constraint of an agent i in period 0:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) e_t^i(s^t). \tag{31}$$

- *Market clearing* has to be sustained at all dates and possible history of events!

$$c_t^1(s^t) + c_t^2(s^t) = e_t^1(s^t) + e_t^2(s^t) \quad \forall t \text{ and } s^t \in S^t. \tag{32}$$

Market Structure: Arrow-Debreu

Definition. An Arrow-Debreu competitive equilibrium is a sequence of allocations $\{c_t^1(s^t), c_t^2(s^t)\}_{t=0,\ s^t \in S^t}^{\infty}$ and prices $\{p_t(s^t)\}_{t=0,\ s^t \in S^t}^{\infty}$ such that:

1. Given the sequence of prices $\{p_t(s^t)\}_{t=0,\ s^t \in S^t}^{\infty}$, for $i = 1, 2$, $\{c_t^1(s^t), c_t^2(s^t\}_{t=0,\ s^t \in S^t}^{\infty}$ is the solution of the problem:

$$\max_{\{c_t^i(s^t) \geq 0\}_{t=0,\ s^t \in S^t}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t)) \tag{33}$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) e_t^i(s^t). \tag{34}$$

2. The goods market is in equilibrium (*feasibility*):

$$c_t^1(s^t) + c_t^2(s^t) = e_t^1(s^t) + e_t^2(s^t) \quad \forall t \text{ and } s^t \in S^t. \tag{35}$$

Market Structure: Arrow-Debreu

- Solution for an arbitrary agent:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t)) + \lambda^i \left(\sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) \left[e_t^i(s^t) - c_t^i(s^t) \right] \right) \tag{36}$$

- And the FOCs...

$$\beta^t \pi(s^t) u'(c_t^i(s^t)) = \lambda^i p_t(s^t) \quad \forall t,\ s^t,\ i$$

- Note that by substituting with λ, the agent equalizes the marginal utility across different states of nature.

$$\beta^t \frac{\pi(s^t)}{\pi(s_0)} \frac{u'(c_t^i(s^t))}{u'(c_0^i(s_0))} = \frac{p_t(s^t)}{p_0(s_0)} \quad \forall t,\ s^t,\ i$$

Market Structure: Arrow-Debreu

- The ratio of marginal utility between agents is constant across all t and s^t:

$$\frac{u'(c_t^2(s^t))}{u'(c_t^1(s^t))} = \frac{u'(c_0^2(s_0))}{u'(c_0^1(s_0))} \quad \forall t,\ s^t$$

- Example with u CRRA:

$$\left(\frac{c_t^2(s^t)}{c_t^1(s^t)}\right)^{-\sigma} = \left(\frac{c_0^2(s_0)}{c_0^1(s_0)}\right)^{-\sigma} \quad \forall t,\ s^t$$

  $\Rightarrow$ The consumption ratio between two agents is constant across all t and s^t.

- Given the resource constraint: $c_t^1(s^t) + c_t^2(s^t) = e_t^1(s^t) + e_t^2(s^t) = e_t(s^t)$: an agent consumes a constant fraction θ^i of the aggregate endowment $e_t(s^t)$.

Market Structure: Arrow-Debreu

- There is **perfect risk sharing** between agents! Consumption fluctuations are given by fluctuations in aggregate income, not individual income.
- The competitive allocation does not depend on the history of events s^t or the distribution of realized endowments (trades are negotiated in period 0).
- Note that we need to assume perfect information and that contracts are enforceable (*full enforcement*).

Market Structure: Arrow-Debreu

- Solving for prices, we use optimality + resource constraint:

$$
\begin{aligned}
p_t(s^t) = & \quad \beta^t \frac{\pi(s^t)}{\pi(s_0)} \left(\frac{c_t^i(s^t)}{c_0^i(s_0)} \right)^{-\sigma} \\
= & \quad \beta^t \frac{\pi(s^t)}{\pi(s_0)} \left(\frac{e_t(s^t)}{e_0(s_0)} \right)^{-\sigma}
\end{aligned}
$$

- That is, the "price" of consumption in a state of nature s^t depends on the probability that this state is realized and the amount of aggregate wealth ($e_t(s^t)$).

- The insurance price in a period of "lean times" is high since no agent wants to distribute their endowments.

Market Structure: Sequential Markets

- Let's define a sequential market structure. In every period, markets open, and trades take place.
- For the equivalence between Arrow-Debreu and sequential markets with uncertainty, we need to deliver one unit of consumption in **all states of nature**.
- The agent can buy a contract at the price of $q_t(s_{t+1}, s^t)$ in period t and history s^t, which delivers one unit of consumption in the next period and state s_{t+1}, for each event s_{t+1}.
- The agent can, in period t, fully protect against any event that will occur in $t+1$ by buying a contract for each s_{t+1}.
- These financial instruments are known as: **Arrow securities**.
- In the case where it is possible to trade *Arrow securities* in all periods and states of nature, Arrow (1964) shows that we can trade *goods* between different t and s^t, that is, we have **complete markets**.

Market Structure: Sequential Markets

- Define $a_{t+1}(s_{t+1}, s^t)$ as the quantity of Arrow securities bought by agents in period t.
- The budget constraint of an arbitrary agent i in t and s^t:

$$c_t^i(s^t) + \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) q(s_{t+1}, s^t) \leq e_t^i(s^t) + a_t^i(s^t)$$

- Note that agents buy Arrow securities in t for all contingencies $s_{t+1} \in S$, but once s_{t+1} is realized, the position of $t+1$ is only $a_{t+1}(s_{t+1}, s^t)$ corresponding to the realized state.
- The Arrow securities market needs to clear at zero for all periods and events.

Market Structure: Sequential Markets

Definition. A competitive equilibrium with Sequential Markets is a sequence of allocations $\{c_t^i(s^t), a_{t+1}^i(s_{t+1}, s^t),\}_{t=0,\ i=1,2,\ s^t\in S^t}^{\infty}$ and prices $\{q(s_{t+1}, s^t)\}_{t=0,\ s^t\in S^t}^{\infty}$ such that:

1. Given the sequence of prices $\{q(s_{t+1}, s^t)\}_{t=0,\ s^t\in S^t}^{\infty}$, for $i = 1, 2$, $\{c_t^i(s^t), a_{t+1}^i(s_{t+1}, s^t),\}_{t=0,\ i=1,2,\ s^t\in S^t}^{\infty}$ is the solution of the problem:

$$\max_{\{c_t^i>0,\ a_{t+1}^i(s_{t+1},s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t \pi(s^t) u(c_t(s^t))$$

$$s.t. \quad c_t^i(s^t) + \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) q_t(s_{t+1}, s^t) \le e_t^i(s^t) + a_t^i(s^t) \quad \forall t,\ s^t$$

$$a_{t+1}^i(s_{t+1}, s^t) \ge -\overline{A}^i \quad \forall t,\ s^t; \quad a_0^i \text{ dado.}$$

2. The goods and asset markets are in equilibrium:

$$c_t^1(s^t) + c_t^2(s^t) = e_t^1(s^t) + e_t^2(s^t) \quad \forall t \text{ and } s^t \in S^t$$

$$a_{t+1}^1(s_{t+1}, s^t) + a_{t+1}^2(s_{t+1}, s^t) = 0 \quad \forall t\ s^t \in S^t \text{ and } s_{t+1} \in S$$

Market Structure: Sequential Markets

- Solution for an arbitrary agent:

$$\mathcal{L} = \sum_{t=0}^{\infty} \left(\sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t)) + \ldots \right.$$

$$\left. \ldots \sum_{s^t \in S^t} \lambda_t^i(s^t) \left[e_t^i(s^t) + a_t^i(s^t) - c_t^i(s^t) - \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) q_t(s_{t+1}, s^t) \right] \right)$$

- The optimality conditions, where $\lambda_{t+1}^i(s_{t+1}, s^t)$ is the multiplier for s_{t+1} given a history s^t:

$$\beta^t \pi(s^t) u'(c_t^i(s^t)) = \lambda_t^i(s^t) \quad \forall t,\ s^t,\ i$$

$$\lambda_t^i(s^t) q_t(s_{t+1}, s^t) = \lambda_{t+1}^i(s_{t+1}, s^t)$$

- Note the equivalence between Arrow-Debreu and sequential when:

$$q_t(s_{t+1}, s^t) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)}$$

Market Structure: Sequential Markets

- In other words, the *pricing kernel* is:

$$q_t(s_{t+1}, s^t) = \beta \frac{u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))} \pi(s^{t+1}|s^t)$$

  where $\pi(s^{t+1}|s^t) = \pi(s_{t+1}, s^t)/\pi(s^t)$.

- The price of **one** Arrow security associated with the state s_{t+1}. Remember that Arrow securities only pay off in one state of nature (in the others, they pay 0).
- The *pricing kernel* is widely used in macro-finance, and from it, we can price various assets.

Market Structure: Sequential Markets

$$\underbrace{q_t(s_{t+1}, s^t)}_{\text{price of the security that pays in state } s_{t+1}} = \underbrace{\beta \frac{u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))}}_{\text{ratio of mg. util. received in state } s_{t+1}} \underbrace{\pi(s^{t+1}|s^t)}_{\text{prob. that state } s_{t+1} \text{ happens}}$$

- Higher probability increases the price of the security.
- Higher mg. utility in s_{t+1} increases the price of the security.

Pricing an Asset

- It is useful to price assets in terms of real returns. Define the one-period realized real return of an asset j between s^t and s^{t+1}:

$$R^j_{t+1}(s^{t+1}) = \frac{P^j_{t+1}(s^{t+1}) + d^j_{t+1}(s^{t+1})}{P^j_t(s^t)}$$

  where $P^j_t(s^t)$ and $d^j_t(s^t)$ is the price of the asset j, in time t and state s^t.

- An arrow security (pays dividend $= 1$ in state s^{t+1} and nothing else in other states nor the future), has gross returns of:

$$R^A_{t+1}(s^{t+1}) = \frac{0+1}{q(s_{t+1}, s^t)} = \frac{1}{q(s_{t+1}, s^t)}$$

Example: Price of Risk-free Bond

- What is the price of a risk-free bond that always pay 1 in the next period (and nothing afterwise)?

$$R_{t+1} = \frac{0+1}{P_t^{\text{risk free}}(s^t)}$$

- The price of the risk free is equivalent of equivalent of having all the possibilities arrow securities:

$$P_t^{\text{risk free}}(s^t) = \sum_{s_{t+1}|s^t} q_t(s_{t+1}, s^t)$$

Example: Price of Risk-free Bond

- Thus, the return of a risk-free bond (non-contingent on the state):

$$\sum_{s_{t+1}|s^t} q_t(s_{t+1}, s^t) = R_{t+1}^{-1}.$$

- In other words, the price of **full consumption insurance** in t is the sum of the prices of Arrow securities associated with all events s_{t+1}:

$$\sum_{s_{t+1}|s^t} q_t(s_{t+1}, s^t) = \beta \sum_{s_{t+1}|s^t} \frac{u'(c_{t+1}^i(s^{t+1}))}{u'(c_t^i(s^t))} \pi(s^{t+1}|s^t)$$

Example: Price of Risk-free Bond

- Note that $\sum_{s_{t+1}} \pi(s^{t+1}|s^t)u'(c^i_{t+1}(s^{t+1})) = \mathbb{E}_t\left[u'(c^i_{t+1})\right]$ is the conditional expected marginal utility of consumption given information in t.

- Substituting R^{-1}_{t+1} and the conditional expectation, we can rewrite the Euler Equation:

$$u'(c^i_t(s^t)) = \beta R_{t+1}\mathbb{E}_t[u'(c^i_{t+1}(s^{t+1})] \quad \forall t,\ s^t.$$

- This is the Euler equation you will encounter most of the time.

Taking Stock

- How to solve a **Dynamic General Equilibrium** model?
  1. Describe the economy's environment;
  2. Solve the agents' problem;
  3. Specify the equilibrium conditions;
  4. Describe the competitive equilibrium.

- How to use the **Welfare Theorems** to solve the model?
  - Given certain conditions, the solution of the **Central Planner** is equivalent to the decentralized equilibrium.
  - In this case, we also know that the equilibrium is Pareto efficient.

- We also have seen that if we have Arrow securities available in all periods and state of nature, markets are complete and the solution of an Arrow-Debreu structure is equivalent to the sequentials market.