PhD in Business Economics - Insper
Macroeconomics 1
Professor: Tomás Martinez
Midterm 2025

Instructions:

- Write your name on the first page of the exam.
- The exam consists of 3 questions totaling 100 points.
- The duration of the exam is 2 hours.
- It is allowed to consult any material. It is not allowed to communicate with other individuals.
- Keep the exam organized and the pages numbered.
- If something in the question is unclear, write the assumptions you deem necessary to have a well-defined problem and proceed.
- If you get stuck on a specific part of a question, remember that you may consider the result of that part as given and continue to answer the other parts.

Questions

1. **(Endowment Economy with 2 agents, 2 periods, and 2 states – 50).** Consider an exchange economy where two agents live for two periods, $t = 1$ and $t = 2$. In period 1, there is no uncertainty and all endowments are known, but in period 2 there are two possible states of nature: s_1 and s_2.

   The probability that the economy is in state s_1 is given by $\pi(s_1) \in (0, 1)$, while the probability of being in s_2 is $\pi(s_2)$, where $\pi(s_1) + \pi(s_2) = 1$. The utility function for agent $i = 1, 2$ is:

$$\ln(c_1^i) + \beta\pi(s_1)\ln(c_2^i(s_1)) + \beta\pi(s_2)\ln(c_2^i(s_2)),$$

   where $c_2^i(s)$ is the consumption of agent i in period 2 and state s. The endowments in each period and state are:

   - Period 1: agent 1 receives $e_1^1 = \hat{e}$, agent 2 receives nothing, $e_1^2 = 0$.
   - Period 2, state s_1: agent 1 receives $e_2^1(s_1) = \hat{e}$, agent 2 receives nothing, $e_2^2(s_1) = 0$.
   - Period 2, state s_2: agent 1 receives nothing, $e_2^1(s_2) = 0$, agent 2 receives $e_2^2(s_2) = \hat{e}$.

   (a) (15 points) Describe the problem of an arbitrary agent i in an Arrow-Debreu framework (trading in period 0). Find the agent's first-order conditions and derive the relationship between first- and second-period consumption, given prices $p_2(s)$ and the probability of an event $\pi(s)$, where $s \in \{s_1, s_2\}$. Normalize the price of the good in the first period to 1.

   Solution: The problem of agent i is:

$$\max_{\{c_1^i, c_2^i(s_1), c_2^i(s_2)\}} \{\ln(c_1^i) + \beta\pi(s_1)\ln(c_2^i(s_1)) + \beta\pi(s_2)\ln(c_2^i(s_2))\}$$
$$s.t. \quad c_1^i + p_2(s_1)c_2^i(s_1) + p_2(s_2)c_2^i(s_2) \leq e_1^i + p_2(s_1)e_2^i(s_1) + p_2(s_2)e_2^i(s_2),$$

   note that we normalize $p_1 = 1$. The Lagrangian (considering the constraint with equality) is:

$$\mathcal{L} = \ln(c_1^i) + \beta\pi(s_1)\ln(c_2^i(s_1)) + \beta\pi(s_2)\ln(c_2^i(s_2)) + \dots$$
$$\dots\lambda^i(e_1^i + p_2(s_1)e_2^i(s_1) + p_2(s_2)e_2^i(s_2) - c_1^i - p_2(s_1)c_2^i(s_1) - p_2(s_2)c_2^i(s_2))$$

   Taking the FOCs:

$$\frac{1}{c_1^i} = \lambda^i$$
$$\frac{\beta\pi(s_1)}{c_2^i(s_1)} = \lambda^i p_2(s_1)$$
$$\frac{\beta\pi(s_2)}{c_2^i(s_2)} = \lambda^i p_2(s_2).$$

   Substituting the FOCs into λ^i, we obtain a relationship between consumption and relative prices for an arbitrary agent i and state $s \in \{s_1, s_2\}$:

$$\frac{c_2^i(s)}{c_1^i} = \frac{\beta\pi(s)}{p_2(s)}$$

(b) (10 points) Write down the equilibrium conditions of this economy.

Solution: The equilibrium conditions in this economy are market clearing for goods in $t = 1, 2$ and for any state $s \in \{s_1, s_2\}$:

$$c_1^1 + c_1^2 = e_1^1 + e_1^2 = \hat{e}$$
$$c_2^1(s) + c_2^2(s) = e_2^1(s) + e_2^2(s) = \hat{e}$$

Note that in all states of nature, the aggregate endowment in this economy is $\hat{e}$.

(c) (15 points) Find an equation that describes the price in period 2 for an arbitrary state of nature s, $p_2(s)$, as a function of the model's parameters. How does $p_2(s)$ change when the probability $\pi(s)$ of the state increases?

Solution: Summing the consumption relationship $\frac{c_2^i(s)}{c_1^i} = \frac{\beta\pi(s)}{p_2(s)}$ for agents $i = 1, 2$:

$$c_2^1(s) + c_2^2(s) = \frac{\beta\pi(s)}{p_2(s)}(c_1^1 + c_1^2).$$

Substituting the equilibrium conditions:

$$\hat{e} = \frac{\beta\pi(s)}{p_2(s)}\hat{e},$$

we find the price function:

$$p_2(s) = \beta\pi(s),$$

which clearly shows that the price of the final good in state s increases linearly with the probability of that state occurring $\pi(s)$.

Note that unlike what we saw in class, the price does not depend on the endowment. This is because the aggregate endowment is the same across all periods and states of nature.

(d) (10 points) Define $\theta \equiv c_1^1/(c_1^1 + c_1^2)$ as the fraction of agent 1's consumption in period 1 relative to the total. Show that when $\pi(s_1) \to 1$, $\theta \to 1$.

Solution: Consider the budget constraint of agent i:

$$e_1^i + p_2(s_1)e_2^i(s_1) + p_2(s_2)e_2^i(s_2) = c_1^i + p_2(s_1)c_2^i(s_1) + p_2(s_2)c_2^i(s_2).$$

Using $\frac{c_2^i(s)}{c_1^i} = \frac{\beta\pi(s)}{p_2(s)}$, and substituting into $c_2^i(s_1)$ and $c_2^i(s_2)$:

$$e_1^i + p_2(s_1)e_2^i(s_1) + p_2(s_2)e_2^i(s_2) = c_1^i(1 + \beta\pi(s_1) + \beta\pi(s_2)),$$

Using $\pi(s_1) + \pi(s_2) = 1$, and substituting the price equation $p_2(s) = \beta\pi(s)$:

$$e_1^i + \beta\pi(s_1)e_2^i(s_1) + \beta\pi(s_2)e_2^i(s_2) = c_1^i(1 + \beta).$$

Rearranging and using the endowments of each individual agent, we have first-period consumption for agents 1 and 2:

$$\text{Agent 1:} \qquad c_1^1 = \frac{\hat{e}}{1+\beta}(1 + \beta\pi(s_1))$$
$$\text{Agent 2:} \qquad c_1^2 = \frac{\hat{e}}{1+\beta}(\beta\pi(s_2))$$

Using the definition $\theta \equiv c_1^1/(c_1^1 + c_1^2) = c_1^1/\hat{e}$:

$$\theta = \frac{1 + \beta\pi(s_1)}{1+\beta},$$

which clearly shows that when $\pi(s_1) \to 1$, then $\theta \to 1$. Intuitively, when $\pi(s_1)$ tends to 1, the income of agent 1 (considering all states of nature) increases relative to agent 2, because the probability of being in the state in which agent 1 receives the endowment is higher. This is reflected in a higher consumption share.

Note that θ is constant across all periods.

2. **(Transition Dynamics in the Neoclassical Growth Model - 15 points).** Consider the standard Neoclassical growth model in continuous time without technological progress. The solution is characterized by the following system of differential equations:

$$\frac{\dot{c}_t}{c_t} = \frac{(f'(k_t) - \delta - \rho)}{\sigma}$$
$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t,$$

together with an initial positive capital stock, $k_0 > 0$, and the Transversality Condition. Suppose that production follows a Cobb-Douglas function: $f(k_t) = k_t^\alpha$, and the economy is initially in the steady state.

(a) (15 points) Suppose that at time t_0, there is an unexpected, temporary increase in the capital elasticity of production (i.e., the parameter α, with $0 < \alpha < 1$, increases). This change lasts until time t_1, after which the economy reverts to its original parameters. Describe the transition dynamics until the economy reaches the steady state at time t_{ss}. Use a phase diagram and other graphs if necessary.

Solution: Note that the initial and final steady states are the same, since the change is temporary.

In the phase diagram, both curves shift at t_0, but return to their original positions at t_1. The inverted U-shaped $\dot{k} = 0$ curve shifts upward, and the vertical $\dot{c} = 0$ line shifts to the right.

- At t_0, consumption jumps upward, but remains below the temporarily shifted $\dot{k} = 0$ curve.
- Since consumption is below the $\dot{k} = 0$ curve, capital accumulates between t_0 and t_1. At the same time, the $\dot{c} = 0$ line has shifted right, so consumption also rises during this interval. At some point between t_0 and t_1, consumption crosses the $\dot{c} = 0$ line and begins to decline.
- At t_1, both curves return to their original positions. At this point, consumption and capital are above their steady-state levels but lie exactly on the saddle path back to the original steady state.
- From t_1 to t_{ss}, both consumption and capital decline, eventually converging to the steady state.

3. **(Expropriation Decision – 35 points.)** Time is discrete and infinite. Consider a firm with production function $y_t = z_t n_t^{\alpha}$, where productivity $z_t \in [0, \infty)$ follows a first-order Markov process with conditional density $f(z_{t+1}|z_t)$, and $\alpha \in (0, 1)$. The firm chooses the number of workers n_t to hire at a fixed wage $w > 0$ over time. The firm sells its product at price 1 (y_t is the *numeraire* of the economy).

There is a government that can expropriate the firm in each period t. Once the firm is expropriated, the government produces with the technology:

$$y_t^g = \theta z_t (n_t^g)^{\alpha},$$

where the parameter $\theta \in (0, 1)$ captures the fact that the government is less efficient than the firm in organizing production. The government only cares about the discounted cash flows generated by an expropriated firm for its budget. That is, the government values an expropriated firm as:

$$\sum_{t=0}^{\infty} \beta^t \pi_t^g = \sum_{t=0}^{\infty} \beta^t (y_t^g - n_t^g w),$$

where $\beta \in (0, 1)$.

In each period, the government has the option to expropriate the firm, which imposes a cost C at time t (due to loss of international reputation, political protests, etc), but no additional costs in the future. Once expropriated, the government controls the firm forever.

(a) (10 points) Find the optimal labor demand for the firm (private sector), n_t^*, and for the government, n_t^{g*}. In which case is labor demand higher?

Solution: The decision is static and can be solved period by period. The government's problem is:

$$\pi_t^g = \max_{n_t^g} \{\theta z_t (n_t^g)^{\alpha} - w n_t^g\}$$

Taking the first-order condition and solving for n_t^{g*}:

$$n_t^{g*} = \left(\frac{\alpha \theta z_t}{w}\right)^{\frac{1}{1-\alpha}} \quad \forall t,$$

The optimal profit (not required, but useful) is:

$$\pi_t^g = (1-\alpha)\frac{(\theta z_t)^{\frac{1}{1-\alpha}}}{w^{\frac{\alpha}{1-\alpha}}} = \pi^g(z). \tag{1}$$

The private firm's solution is the same with $\theta = 1$. Therefore, $n_t^* > n_t^{g*}$ and $\pi_t > \pi_t^g$ (for a given z).

(b) (10 points) Find the Bellman equation that characterizes the value W of an expropriated firm to the government (excluding the expropriation cost C).

Solution: Note that both n_t^{g*} and π_t^{g*} depend on time only through z_t, so knowing z allows us to compute labor demand and profit. Therefore, the Bellman equation depends only on z:

$$
\begin{aligned}
W(z) &= \max_{n^g} \{\theta z_t (n_t^g)^\alpha - w n_t^g + \beta \mathbb{E}[W(z')|z]\} \\
&= \max_{n^g} \{\theta z_t (n_t^g)^\alpha - w n_t^g\} + \beta \mathbb{E}[W(z')|z] \\
&= \pi^g(z) + \beta \int_0^\infty W(z') f(z'|z) dz'
\end{aligned}
$$

(c) (15 points) Write the Bellman equation that characterizes the government's expropriation decision, V. Describe the state(s), control(s), and return function. You may use the value function W from the previous question if you wish.

Solution: State: z. Control: expropriation decision $e \in \{1, 0\}$, where $e = 1$ indicates expropriation. Return function $F(z, e) = \pi^g(z) - C$ if $e = 1$ and $F(z, e) = 0$ if $e = 0$.

The Bellman equation is:

$$
V(z) = \max_{e \in \{1,0\}} \{e(W(z) - C) + (1 - e)\beta \mathbb{E}[V(z')|z]\} = \max \{W(z) - C, \beta \mathbb{E}[V(z')|z]\},
$$

where $\mathbb{E}[V(z')|z] = \int_0^\infty V(z') f(z'|z) dz'$.