PhD in Business Economics - Insper
Macroeconomics 1
Professor: Tomás Martinez
Midterm 2024

Instructions:

- Write your name on the first page of the exam.
- The exam consists of 3 questions totaling 50 points.
- The duration of the exam is 2 hours.
- It is allowed to consult any material. It is not allowed to communicate with other individuals.
- Keep the exam organized and the pages numbered.
- If something in the question is unclear, write the assumptions you deem necessary to have a well-defined problem and proceed.
- If you get stuck on a specific part of a question, remember that you may consider the result of that part as given and continue to answer the other parts.

Questions

1. **(ChatGPT in Neoclassical Growth - 20 points).** Consider the neoclassical growth model in continuous time.

   A representative firm produces a single final good using capital and labor that can be consumed or invested. The production function is given by $Y_t = Z_t K_t^{\alpha}(A_t L_t)^{1-\alpha}$. The population grows at rate n and the technology biased towards labor, A_t, grows at rate g:

$$\frac{\dot{A}_t}{A_t} = g \qquad \& \qquad \frac{\dot{L}_t}{L_t} = n.$$

   Z_t does not follow a growth trend, but drastic innovations (such as ChatGPT) can alter its value from one period to another. Assume: $L_0 = 1, A_0 = 1, Z_0 = 1, K_0 > 0$ and define the variables per efficient unit of labor as: $x = \frac{X}{AL}$.

   The utility of the representative family is:

$$\max_{c_t \geq 0} \int_0^{\infty} e^{-(\rho-n-g(1-\sigma))t} \frac{c_t^{1-\sigma}}{1-\sigma} dt,$$

   where $\rho > n + g(1-\sigma)$. The resource constraint in efficient units of labor is given by

$$\dot{k}_t = Z_t k_t^{\alpha} - c_t - (n+\delta+g)k_t \qquad \forall t.$$

   Suppose that the transversality condition is satisfied.

   (a) (7 points) Find a system of differential equations that characterize the planner's solution for this economy.

   Solution: The (discounted) Hamiltonian is:

$$\hat{H}(k_t, c_t, \mu_t) = u(c_t) + \mu_t(Z_t k_t^{\alpha} - c_t - (n+g+\delta)k_t)$$

   The necessary conditions (together with the TVC and the law of motion) at every t:

$$\begin{aligned} u'(c_t) &= \mu_t \\ \mu_t[\alpha Z_t k_t^{\alpha-1} - n - g - \delta] &= -\dot{\mu}_t + (\rho - n - g(1-\sigma))\mu_t \end{aligned}$$

   Euler Equation:

$$\frac{u''(c_t)\dot{c}_t}{u'(c_t)} = -(\alpha Z_t k_t^{\alpha-1} - \delta - \rho - g\sigma) \qquad \forall t.$$

   Using the functional form of the utility function, we find the EE:

$$\frac{\dot{c}_t}{c_t} = \frac{\alpha Z_t k_t^{\alpha-1} - \delta - \rho - g\sigma}{\sigma} \qquad \forall t.$$

   To complete the system of differential equations, we have the resource constraint per efficient unit of labor (in addition to TVC and given the initial condition k_0):

$$\dot{k}_t = Z_t k_t^{\alpha} - c_t - (n+g+\delta)k_t \qquad \forall t.$$

(b) (6 points) Suppose that Z_t in steady state is given by Z. Derive the equations that characterize the steady state and draw the phase diagram.

Solution: The steady state can be found by solving the system:

$$\alpha Z k_{ss}^{\alpha-1} = \delta + \sigma g + \rho \qquad \Leftrightarrow k_{ss} = \left(\frac{\alpha Z}{\rho + \sigma g + \delta}\right)^{1/(1-\alpha)}$$
$$c_{ss} = Z k_{ss}^{\alpha} - (n + g + \delta) k_{ss} \Leftrightarrow c_{ss} = \left(\frac{\alpha Z}{\rho + \delta + \sigma g}\right)^{1/(1-\alpha)} \left(\frac{\delta + \rho + \sigma g - \alpha(n + g + \delta)}{\alpha}\right)$$

The phase diagram is the same as seen in class.

(c) (7 points) Suppose the economy is in steady state. In period t_0, a new artificial intelligence technology is developed. This innovation makes the economy more productive in the short term but returns to the original level in the long term.[1]

From the planner's point of view, at instant t_0 a change from Z to $\hat{Z}$ occurs, where $\hat{Z} > Z$, but at t_1, $\hat{Z}$ moves back to Z. Describe the transition dynamics until the economy reaches the steady state at t_{ss} (use the phase diagram and other graphs if necessary).

Solution: Note that the initial and final steady states are the same, since the change is temporary.

In the phase diagram, both curves shift at t_0, but return to their original state at t_1. The inverted u-shaped curve shifts upwards and the line determining capital in the steady state shifts to the right.

- At t_0, consumption jumps upwards but below the $\dot{k}$ curve, which momentarily shifted upwards.
- As consumption is below the $\dot{k}$ curve, the economy accumulates capital between t_0 and t_1. On the other hand, the $\dot{c}$ line has shifted to the right, so consumption increases between t_0 and t_1. At some point between t_0 and t_1, consumption crosses the $\dot{c}$ line and consumption starts to decrease.
- At t_1, both curves return to the original point. At this moment, consumption and capital will be above the steady state and exactly on the saddle path to the old steady state.
- From t_1 to t_{ss}, both consumption and capital will decrease and eventually reach the steady state.

[1] We can rationalize this by imaging artificial intelligence reduces the intelligence of new generations and therefore the total factor productivity returns to the original level.

2. **(Human Capital Investment - 11 points).** Consider an agent who accumulates human capital $h_t \geq 0$ according to the technology $h_{t+1} = g(h_t) + x_t$, where $x_t \geq 0$ is a monetary investment in human capital and $g(.)$ is a continuous, bounded, and strictly increasing function. The agent cannot save and receives labor income wh_t (where the wage $w > 0$ is fixed). The agent derives utility according to:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u(.)$ is continuous, bounded, strictly increasing, and concave. The agent's budget constraint in all periods is:

$$c_t + x_t \leq wh_t,$$

and $h_0 > 0$ is given.

(a) (6 points) Describe the problem in dynamic programming form: write the Bellman equation and clearly specify the state, control, feasible set, and the return function.

Solution: The budget constraint is sustained with equality since utility is strictly increasing. Note that we can write the problem with different controls (x, h', or even c). It's necessary to be careful and write the state law of motion correctly if the state is not the same as the control.

$$V(h) = \max_{x \in [0,wh]} \{u(wh - x) + \beta V(g(h) + x)\}, \text{ or}$$
$$V(h) = \max_{h' \in [g(h),g(h)+wh]} \{u(wh + g(h) - h') + \beta V(h')\}$$

(b) (5 points) Define an operator T on $C(X)$ (set of continuous and bounded functions with the supremum norm) that allow us to find the value function. Show that the operator T is a contraction (you can assume that T is a map from $C(X)$ to $C(X)$).

Solution: Operator:

$$TV(h) = \max_{x \in [0,wh]} \{u(wh - x) + \beta V(g(h) + x)\}$$

We need to check if the operator satisfies Blackwell's sufficient conditions: discounting and monotonicity. Follow the steps done in the slides and you can easily see that the operator satisfies both properties.

In general, we must also check that the operator maps $C(X)$ to $C(X)$. The question mentions that you can assume it so you do not need to do that. If you want to check it, you should apply the Maximum Theorem of Bergé. Argue that the operator satisfies the assumptions of the theorem:

- $\Gamma(h) = [0, wh]$ or $\Gamma(h) = [g(h), g(h)+wh]$ are non-empty, continuous correspondences with compact (and convex) values.
- Suppose $V \in C(X)$, then $f(h, x) \equiv u(wh - x) + \beta V(g(h) + x)$ or $f(h, h') \equiv u(wh + g(h) - h') + \beta V(h')$ are continuous functions since u, g, and V are continuous, and the sum and composition of continuous functions are also continuous.

Thus, we can apply Bergé and $TV(h)$ is continuous. If V is bounded, as u is bounded and the sum of bounded functions is also bounded, then $TV(h)$ is bounded. Hence, $TV(h) \in C(X)$.

3. **(Consumption and Savings with Endogenous Retirement - 19 points).** Consider the problem of an agent who lives a finite life, decides how much to consume, save, and when to retire. The agent receives utility from consumption and disutility $\gamma > 0$ from working. The utility of this agent is:

$$\mathbb{E}_0 \sum_{t=0}^{T} \beta^t [u(c_t) - \gamma d_t],$$

where d_t is an indicator function that takes the value 1 if the individual is working and 0 if retired, $u(.)$ is continuous, strictly increasing, and concave, and $\beta \in (0, 1)$.

Suppose the agent can save at a gross interest rate, $(1+r) > 1$, and that she/he cannot borrow: $a_{t+1} \geq 0$. The budget constraint is:

$$c_t + a_{t+1} = y_t + a_t(1 + r), \qquad t = 0, 1, 2, ...T,$$

and $a_0 = 0$. The individual's income is given by:

$$y_t = \begin{cases} w_t, & \text{if working } (d_t = 1), \\ p(x_t), & \text{if retired } (d_t = 0), \end{cases}$$

where the wage, w_t, follows a first-order Markov chain, and retirement income, $p(x_t)$, is a function of years worked, x_t, with $p'(x_t) \geq 0$. Note that $x_0 = 0$ and each year worked $x_{t+1} = x_t + 1$, while during retirement $x_{t+1} = x_t$.

The timing of the problem is as follows: at the beginning of the period the agent observes the realization of the wage, and then decides whether to retire or not. After the retirement decision, she/he receives pension/wage and decides how much to consume/save. If she/he decides to retire, the agent CANNOT return to work.

(a) (6 points) Define the value for an individual who has just retired at age t as $V_t^R(a)$. Find an equation that determines this value.

Solution: Define $V_t^R(a)$ as the value of an individual who decided to retire at age t:

$$V_t^R(a) = \max_{\{a_{t+s+1}\}_{s=0}^{T}} \sum_{s=0}^{T} \beta^s u(p(t) + a_{t+s}(1 + r) - a_{t+s+1})$$

Note that $p(t)$ is fixed after the retirement. Although not ideal, it can also be written in dynamic programming form. The important thing is that $p(x)$ is fixed after the retirement decision, so it is necessary to separate the age t from the years worked x.

Note: It is not necessary to do this in this problem, but in practice, when we have the functional form $u()$, we can use the Euler equation along with $a_{T+1} = 0$ and find an analytical form for $V_t^R(a)$.

(b) (7 points) Write the Bellman equation of the agent (before the retirement decision) and clearly specify the state and controls (and their constraints).

Solution:

- State: t, a, and w. It is not necessary to use x as a state, since it is exactly equal to t. If the timing of the problem were different, we could write it with the cash-on-hand formulation.[2]
- Controls: $d = \{0, 1\}$ (retirement decision) and $a' \in [0, w + a(1 + r)]$ (if working).
- Define $V_t^W(a, w)$ as the value after the decision to continue working. The Bellman equation:

$$\begin{aligned} V_t(a, w) &= \max_{d \in \{0,1\})} \{d_t V_t^W(a, w) + (1 - d_t) V_t^{Ap}(a)\} = \max\{V_t^W(a, w), V_t^R(a)\} \\ V_t^W(a, w) &= \max_{a' \in [0, w + a(1+r)]} \{u(w + a(1 + r) - a') - \gamma + \beta \mathbb{E}[V_{t+1}(a', w')|w]\} \end{aligned}$$

(c) (6 points) Describe an algorithm to find the agent's Bellman equation. Clearly specify the steps of the algorithm.

Solution: Since it is a finite time problem, we can solve it using backward induction knowing that $a_{T+1} = 0$. Since the problem involves a discrete choice (retirement), it is necessary to solve both value functions. It is always ideal to start with the absorbing state (in this case, retirement).

In question (a), we saw that the retirement value is:

$$V_t^R(a) = \max_{\{a_{t+s+1}\}_{s=0}^T} \sum_{s=0}^{T} \beta^s u(p(t) + a_{t+s}(1 + r) - a_{t+s+1})$$

Since the income $p(t)$ is determined at the time of retirement and is constant. The retirement value (at each age t) can be computed using backward induction or via Euler equation.

Once we have $V_t^R(a)$ for all ages, we can compute $V_t(a, w)$ backwards. Starting from the last period:

$$\begin{aligned} V_T(a, w) &= \max\{V_T^W(a, w), V_T^R(a)\} \\ V_T(a, w) &= \max\{u(w + a(1 + r)) - \gamma, u(p(T) + a(1 + r))\}. \end{aligned}$$

With $V_T(a, w)$ we can use the Markov chain of w and compute the expectation $\mathbb{E}[V_T(a', w')|w] = \int V_T(a', w') f(w'|w) dw'$. With this, the value of working in $T - 1$:

$$V_{T-1}^W(a, w) = \max_{a' \in [0, w + a(1+r)]} \{u(w + a(1 + r) - a') - \gamma + \beta \mathbb{E}[V_T(a', w')|w]\},$$

and using $V_{T-1}^R(a)$ we can compute $V_{T-1}(a, w)$:

$$V_{T-1}(a, w) = \max\{V_{T-1}^W(a, w), V_{T-1}^R(a)\}.$$

Using this iterative process, we can compute $V_{T-1}(a, w),\ V_{T-2}(a, w), \ldots V_0(a, w)$.

[2] The problem was based on Iskhakov et al (2017, Quantitative Economics).