Advanced Macroeconomics

HH Heterogeneity: Transition Dynamics and Aggregate Fluctuations

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Introduction

- At this point, we have only focus on the **Stationary Equilibrium**.
- Many questions involve solving the model beyond the Steady-State Stationary Equilibrium.
- **Aggregate Uncertainty:**
  - How the Aiyagari economy reacts to aggregate shocks.
  - Does heterogeneity matters to the business cycles?
- **Transitions Dynamics:**
  - How long it takes to the economy reach a new steady state after an economic reform.
  - How to compute the transition from one steady state to another.

References

- Boppart, Krusell and Mitman (2018, JEDC)*: Intuitive paper on how transition dynamics can be used to simulate aggregate shocks (+ history about the MIT shocks).
- Auclert, Bardóczy, Rognlie and Straub (2021, ECTA)*: State-of-the-art method to solve HA models with aggregate uncertainty.
- Krusell and Smith (1998, JPE): original paper outlining the famous algorithm.
- Krueger, Mitman and Perri (2016, Handbook of Macro): Application of the model to the great recession.
- Heer and Maussner (2009): Ch. 8 and 10; Fehr and Kindermann (2019): Ch. 11: Textbook treatment of the computational methods.
- Algan et al (2014, Handbook of Computational Economics): Entire handbook on how to solve HA economies with aggregate uncertainty. See also their special edition on the JEDC.

Aiyagari + Aggregate Uncertainty

- We are going to focus in the simplest version of the Aiyagari model with aggregate uncertainty.

- The only modification is an aggregate TFP shock in the production funcion:

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$$
$$\log Z_t = \rho_z \log Z_{t-1} + \sigma_z \varepsilon_t$$

- Sometimes this is known as the **Krusell-Smith** economy.

Aiyagari + Aggregate Uncertainty

- Prices are allowed to vary over the cycles.
- To write in recursive form, we also include the aggregate variables as a state in the HH problem.
  - Individual state: (a, s), aggregate state: (Z, λ).

$$
\begin{aligned}
V(a, s; Z, \lambda) = & \max_{c,\ a' \geq 0} \{u(c) + \beta \mathbb{E}[V(a', s'; Z', \lambda') | s, Z]\} \\
\text{s.t} \qquad & c + a' = w(Z, \lambda) \exp\{s\} + (1 + r(Z, \lambda) - \delta) a \\
& \lambda' = H(Z, \lambda, Z')
\end{aligned}
$$

  - Note the dependence of prices on the distribution.
  - The function H is the law of motion/forecasting function of the distribution.

Equilibrium

- Prices are given by FOCs of firm's problem:

$$w(Z,\lambda) = Z\alpha \left(\frac{K(Z,\lambda)}{N(Z)}\right)^{1-\alpha} \quad \text{and} \quad r(Z,\lambda) = Z(1-\alpha)\left(\frac{N(Z)}{K(Z,\lambda)}\right)^{\alpha}$$

  where aggregate employment $N(Z)$ is given by the distributions of the Markov process (which may depend on Z).

- Asset market clears:

$$K(Z,\lambda) = \int a d\lambda$$

- The distribution evolves according to the function: $\lambda' = H(Z,\lambda,Z')$. In equilibrium, with rational expectation, this function is *consistent* with the individual decisions.

HA with Aggregate Uncertainty

Hard problem to solve:

- Prices are functions of the distribution, so distribution must be part of the state space.
- But the distribution is infinitesimal object with a lot of information.
- Furthermore, agents must forecast the evolution of the distribution to form expectations.
- And the forecast has to be consistent with the individuals decision (i.e., fixed point).

Computational

To solve a heterogeneous agent economy with aggregate uncertainty the main methods are:

- **State-space Methods:**
  - Krusell-Smith (1998, JPE) bounded rationality algorithm.
  - Reiter (2009, JEDC) Method.
- **Sequence-space Methods:**
  - MIT shock (Boppart, Krusell and Mitman, 2018, JEDC).
  - Auclert, Bardóczy, Rognlie and Straub (2021, ECTA) sequence space Jacobian.
- There are others/variations of algorithms and evolutions of the original ones. Check Algan et al (2014).

Krusell & Smith (1998) Bounded Rationality

- References: Krusell-Smith's original paper is easy to follow. Check also Nakajima's notes.
- Because prices are allowed to vary over the cycles and they are needed for the household problem: the aggregate state, (Z, λ), is part of the state of the HH.
  - **Problem:** the distribution, λ, is a high-dimensional object and the state space increases substantially.
- **Krussel & Smith (1998)**: instead of using the entire distribution, just use some moments of the distribution:
  - Households are "boundedly rational" on how the distribution evolves.
  - In this class of models, the **mean** (first moment) is enough to correctly forecast prices:

$$\lambda' = H(Z, \lambda, Z') \qquad \Rightarrow \qquad K' = H(Z, K, Z') \tag{1}$$

Krusell & Smith (1998)

- Substitute λ by K. Example:

$$
\begin{aligned}
V(a,s;Z,K) = \max_{c,\ a'\geq 0} \{u(c) + \beta\mathbb{E}[V(a',s';Z',K')|s,Z]\} \\
\text{s.t} \qquad c + a' = w(Z,K)\exp\{s\} + (1 + r(Z,K) - \delta)a \\
K' = H(Z,K,Z')
\end{aligned}
$$

- **Intuition:** the mean of λ works well to forecast prices because the savings policy function is approximately linear.
  - The curvature of the policy function is close to the borrowing constraint, but these agents hold little wealth and thus do not matter to the aggregate.

- For more complex models, one may need higher moments.

Krusell-Smith Algorithm

- Suppose Z is a two-state Markov-Chain (recession and boom).

- Approximate the function forecasting function $H()$ with a log-linear form:

$$\log K' = a_l + b_l \log K \quad \text{if } Z = Z_l$$
$$\log K' = a_h + b_h \log K \quad \text{if } Z = Z_h$$

- We have to find the parameters: (a_l, a_h, b_l, b_h).

- As any continuous state, we must discretize K so we must interpolate when applying the function above.

Krusell-Smith Algorithm

Discretize the state space: (a, s, K, Z). Recover the prices $r(K, Z)$ and $w(K, Z)$ for each state space using the firm's problem.

(i) Guess the parameters of the forecast function: $(a_l^0, a_h^0, b_l^0, b_h^0)$.

(ii) Given $(a_l^0, a_h^0, b_l^0, b_h^0)$, solve the Bellman Equation of the HH for all the state space (a, s, K, Z).

(iii) Given the household policy functions, simulate T periods:

- Draw a sequence of Z_t for all T. Guess a initial distribution λ_0.
- Using the policy function and the sequence Z_t, keep updating the distribution λ_t forward.
- Compute the mean of the distribution K_t (and other moments if necessary).
- Drop the first T_0 periods. Now, we have a sequence $\{Z_t, K_t\}_{t=T_0}^{T}$.

Krusell-Smith Algorithm

(iv) Using the sequence $\{Z_t, K_t\}_{t=T_0}^{T}$, run a linear regression and recover the new coefficients: $(a_l^1, a_h^1, b_l^1, b_h^1)$.

(v) Check the distance between the guess a^0, b^0 and the new parameters a^1, b^1. If it is smaller than tol, we are done. Otherwise, update the guess and start again:

$$a^0 = d\ a^0 + (1-d)a^1$$
$$b^0 = d\ b^0 + (1-d)b^1$$

where $d \in (0,1)$ is a damping parameter.

Krusell-Smith Algorithm: Issues

- After you finish, you must check the R^2 of the forecast regression. If the R^2 is low, you must add more moments or change the function form.
  - In Krusell-Smith, $R^2 = 0.999$, so the perceived law of motion of K is very close to the actual law of motion.

- Poor initial guesses might not converge. One good guess is $a = \log K_{ss}$ and $b = 0$.

- **Good:** KS captures potential non-linearities and large shocks. For instance, asymmetries between the boom and the recession; uncertainty shocks; etc.

- **Bad:** KS can be inaccurate if there are explicitly distributional channels coming from the top of the wealth distribution. Potentially very slow.

Krussell-Smith: State of the Art

- If you need to solve a HA model using a truly global method, the state-of-the-art nowadays is to use **Deep learning/machine learning**:

- See Fernandéz-Villaverde, Hurtado & Nuño (ECTA, 2023); Azinovic et al (IER, 2022); Maliar et al (2021); and other papers by Fernandéz-Villaverde and Galo Nuño.

Reiter Method: Projection + Pertubation

- **Perturbation Methods**:
  - Generalization of the well-known linearization around the steady state.
  - Often used to solve DSGE/representative agent models.
  - They tend to be fast, but require derivatives and some stability conditions (Blanchard-Kahn).
- Standard software (i.e., dynare) uses this method.
- Reiter (2009) propose to solve for the stationary equilibrium using global methods (projection methods), and then use perturbation methods to solve for the aggregate shock.
- If you need a refresher on Pertubation methods, check Fernandez-Villaverde's notes.

Reiter's Method

- We can write the solution of DSGE models as a nonlinear system of difference equations:

$$E_t F(x_t, x_{t+1}, y_t, y_{t+1}) = 0 \quad (2)$$

  where x is the vector of predetermined variables (state), y is nonpredetermined variables (control).

- Then, we can linearize the system (either numerically or analytically) and use methods to solve the linear system of difference equations:
  - Blanchard and Kahn (1980); Uhlig (1999); Sims (2000); Rendahl (2018).

Reiter's Method

- **Example**: Stochastic Neoclassical Growth model

$$E_t F(x_t, x_{t+1}, y_t, y_{t+1}) = E_t \begin{bmatrix} c_t^{-\gamma} - \beta E_t c_{t+1}^{-\gamma}[\alpha k_{t+1}^{\alpha-1} + 1 - \delta] \\ c_t + k_{t+1} - e^{z_t} k_t^{\alpha} - (1-\delta) k_t \\ z_{t+1} - \rho z_t - \sigma \varepsilon_{t+1} \end{bmatrix} = 0$$

where $x = [k, z]'$ and $y = [c]$.

- First row is the Euler Equation, second is the feasibility constraint, and the last is the stochastic process of the shock.

Reiter's Method

- **Example**: Krusell-Smith economy.

$$E_t F(x_t, x_{t+1}, y_t, y_{t+1}) = E_t \begin{bmatrix} \lambda_{t+1} - \lambda_t \Pi_{g_{a,t}} \\ V_t - (\overline{u}_{g_{a,t}} + \beta \Pi_{g_{a,t}} V_{t+1}) \\ z_{t+1} - \rho z_t - \sigma \varepsilon_{t+1} \\ \text{ED}(g_{a,t}, \lambda_t, z_t, P_t) \end{bmatrix}$$

where $x = [\lambda, z]'$ and $y = [V P]'$.

  - λ is the p.d.f of the distribution;
  - P_t are the prices;
  - ED() is an arbitrary excess demand function (which implicitly includes firm's foc);
  - $\Pi_{g_{a,t}}$ is the transition matrix induced by the optimal policy:

$$g_{a,t} = \arg\max u(a(1+r_t) + w_t s - a') + \beta E_t V_{t+1}(a', s', \lambda', z')$$

Reiter's Method

- Since we discretize both λ and V_t, the first two rows must hold for ALL the idiosyncratic state.
- The number of equations that we need to linearize is exponentially large.
- Linearization is often done using numerical derivatives. Nowadays people use automatic differentiation to do the job.
- Solution (up to first order) has **certainty equivalence**: no precautionary savings because of aggregate risk.
- The method cannot capture nonlinearities or sign asymmetries (again up, to first order).

Reiter's Method: State-of-the-art

- **Good:**
  - Similar to standard methods using in RA-DSGE (some people argue that is possible to do it in Dynare).
  - Easier to do second-order approximations than the sequence-space methods and faster than fully global methods.

- **Bad:**
  - Tend to be quite hard to implement because they require some type of dimensionality reduction to be fast.
  - Numerical derivatives can be unstable when mixed with discretization.

- **State-of-the-art**:
  - Bayer and Luetticke (QE, 2020): The codes are available in their website (Matlab, Python and Julia): https://www.ralphluetticke.com/.
  - Ahn, Kaplan, Moll, Winberry and Wolf (NBER Macro, 2018); Bhandari et al. (2023) - Second and higher-order approx; Winberry (QE, 2018) - HA Firms, but implemented in Dynare; Bilal (2023).

Sequence Space

- Instead of including the aggregate variables in the state-space, we can index everything through time: $\{r_t, w_t, V_t, \lambda_t, ...\}_{t=0}^{T}$.
- Then, we solve the model in the **Sequence space** from $t \in \{0, .., T\}$, where T is a large number.
- For instance, we can simulate an **impulse response function** (IRF), which is just a deterministic transition dynamics between two steady states after an unexpected aggregate shock (a MIT shock).
- **Boppart, Krusell and Mitman (2018)** show that the IRF can be used to compute equilibrium of HA with agg. uncertainty.
  - Solving for the transition dynamics is also useful if you are interested in studying the transition to a new steady state after a change in economic policy.

Transition Dynamics and MIT shocks

- **MIT shock:** an unpredictable shock to the steady-state equilibrium of an economy without shocks.
  - No shocks are expected to ever materialize but nevertheless a shock now occurs!
- We can now analyze the equilibrium transition along a **perfect-foresight path** until the economy reaches the steady state.
- Some argue that **Tom Sargent** coined the term reflecting that some researchers at MIT used the method.
  - For fresh-water economists, a MIT shock is inconsistent with rational expectations!
  - "A shock of probability zero, only at MIT they can get away with that!".

MIT shock

- Suppose a standard Ayiagari in the steady state at $t = 0$. At $t = 1$, the economy receives an (unexpected) TFP aggregate shock:

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$$
$$\log Z_t = \rho_z \log Z_{t-1} + \varepsilon_t$$

  where $\varepsilon_t = 0.01$ if $t = 1$ and $\varepsilon_t = 0$ otherwise.

- If $0 < \rho_z < 1$, when $t \to \infty$, the shock vanishes and we are back to the original steady state.

- Our goal is to solve the **transition dynamics** between the two steady states.
  - Because Z_t varies in the transition, aggregate variables (prices, savings, distribution) change during the transition.

Sequential Equilibrium

- Instead of carrying the aggregate state, we index the Bellman Equation by time t.

$$V_t(a,s) = \max_{c,\ a' \geq 0} \left\{ u(c) + \beta \sum_{s' \in S} \pi(s'|s) V_{t+1}(a's') \right\}$$

$$\text{s.t} \qquad c + a' = w_t s + (1 + r_t - \delta)a$$

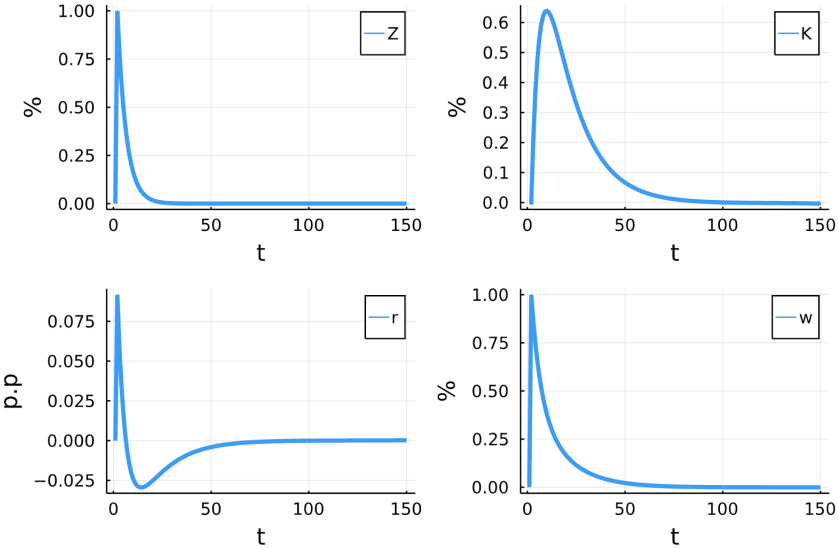
- The distribution follows the L.O.M: $\lambda_{t+1} = \Pi_{g_{a,t}} \lambda_t \quad \forall t$.

- The asset market must clear for all t:

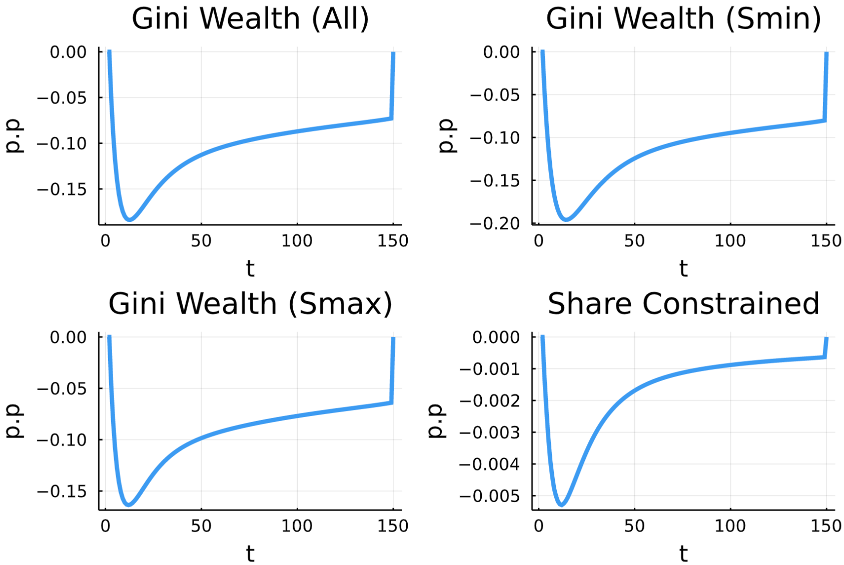
$$\int_{A \times S} a d\lambda_t(a,s;r_t) \equiv A_t(r_t) = K_t(r_t)$$

both the distribution, $\lambda_t(a,s)$, and the aggregate capital, K_t, are indexed by t.

IRF: Standard Aiyagari Economy



IRF: Standard Aiyagari Economy



Transition Dynamics between Steady States

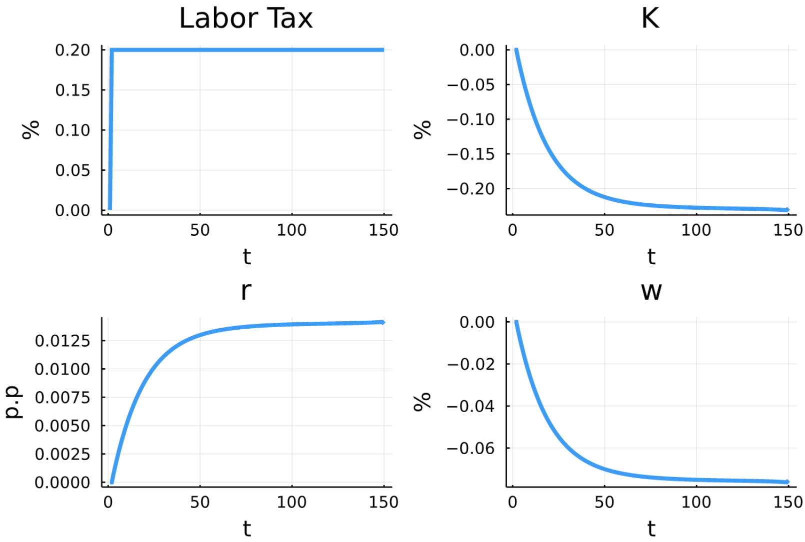
- The method is useful to compute transition between different steady states.
- **Example**: Suppose a labor tax, τ_l, that is used to finance a lump-sum transfer, T_t. The budget constraint:

$$c + a' = w_t s(1 - \tau_l) + (1 + r_t - \delta)a + T_t.$$

  The government runs a balanced budget: $T_t = \tau_l w_t L$.

- Suppose the economy is in the SS with $\tau_l = 0$. At $t = 1$, the government decides to raise the tax rate: $\tau_l = 0.2$ (there are no aggregate shocks).
- How long does the economy take to reach the new steady state?

Transition to New SS: Labor Tax



Algorithm

(i) Solve for the initial and the final steady state. Select a large number of periods T.

(ii) Guess a path of $\{K_t^g\}_{t=2}^{T-1}$. K_1 and K_T are given by the initial/final steady state. Recover the prices $\{r_t, w_t\}_{t=2}^{T-1}$ using the firm's problem and the sequence of Z_t.

(iii) Given prices, $\{r_t, w_t\}_{t=2}^{T}$, solve the value function (and policy functions) backwards from $t = T-1, ...2$ starting from the **final steady state value function**.
- Endogenous Grid works well, but careful to use the correct prices!

(iv) Starting from the **initial steady state distribution**, simulate the distribution forward from $t = 1, ..., T-1$ using the policy functions, $g_{a,t}(a, s)$ and the Markov process of s.

Algorithm

(v) Compute aggregate savings (capital) using the distribution for all t: $\{K_t^s\}_{t=2}^{T-1}$.

(vi) Compute the maximum difference between the guess sequence, $\{K_t^g\}$, and the new sequence, $\{K_t^s\}$. If it is smaller than tol, stop. Otherwise, update the guess using the rule:

$$K_t = dK_t^s + (1-d)K_t^g \qquad \text{for } t = 2, .., T-1,$$

where $\lambda \in (0,1)$ is a dampening parameter, and return to (ii).

Algorithm

- The "shooting algorithm" does not have established convergence properties but tends to work well in practice.
- The damp parameter should not be too large, otherwise, it may not converge.
- T has to be large enough to allow the shock to fade out completely. Always check the last transition between times $T-1$ and T.
- A good initial guess is $K_{ss} = K_t$ for all t.
- If labor supply is endogenous you can guess K/L. If you need to find the eq. in other markets you have to guess an additional sequence.

Boppart-Krusell-Mitman (2018)

- Intuitively, the method uses the impulse response function as a sufficient statistic to compute the eq. of the model.
- In theory, dynamic programming says that any aggregate statistic of the model can be computed as a function of the aggregate state: $x(Z, \lambda)$.
- Instead of using aggregate state, we can also write the aggregate stats as a function of past shocks. For example, the aggregate capital at time t is:

$$K_t = K(\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ...),$$

where ε_t is the innovation of the aggregate at time t.

Boppart-Krusell-Mitman (2018)

- If we assume that the model response to the shock is approximately linear, we can write K_t as a linear function of past shocks:

$$K_t = K(\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ...) = \varepsilon_t K(1,0,0,...) + \varepsilon_{t-1} K(0,1,0,...) + \varepsilon_{t-2} K(0,0,1,...) + ...$$

  where $K(0,1,0,...)$ is the (non-linear) response of capital at time t to a shock (scaled to 1) that happened at $t-1$.

- Note that each K is the response of ONLY ONE shock at each point in time.

- In the notation of BKM: $K_0 = K(1,0,0,...)$, $K_1 = K(0,1,0,...)$, etc. Then:

$$K_t = \sum_{s=0}^{\infty} \varepsilon_{t-s} K_s$$

Boppart-Krusell-Mitman (2018)

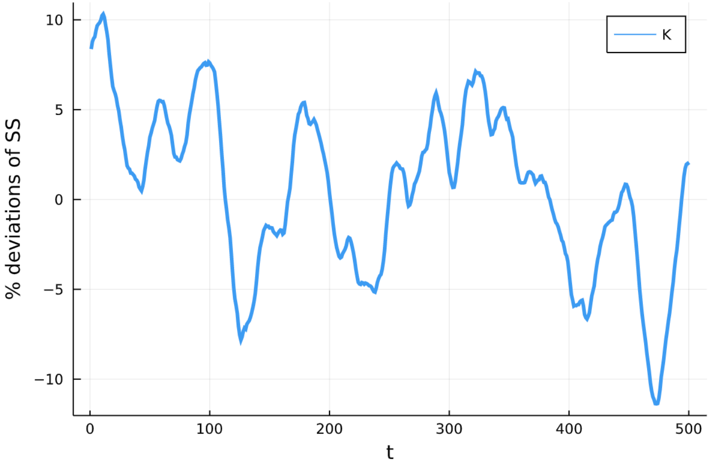
- When we compute an impulse response function to an MIT shock, we get exactly the response of capital to a 1% shock that happened s periods before!
- That is, we have a sequence of K:

$$[K(1,0,0,...),K(0,1,0,...),K(0,0,1,...),...]$$

- We have that for all aggregate statistics of the model: $C,\ A,....$
- To simulate the model, we can simply draw a sequence of shocks ε and use the statistics computed by the impulse response.

Boppart-Krusell-Mitman (2018)

Figure: Simulation of Aggregate Capital using BKM



Boppart-Krusell-Mitman (2018)

- **Good:** It is easy to use. The only thing you need is an impulse response function. You can compute using standard dynamic programming methods.

- It is trivial to add more shocks. Because shocks are linear, you just need to simulate two IRF for each shock. Then, the final effect of the shocks is simply additive.

- **Bad:** If the model is highly non-linear or has sign-dependence it can be a poor approximation.

- As every other linear method, it assumes certainty equivalence. No second-order effects from aggregate risk; It may perform poorly if the shock brings you far from the steady state.

Sequence-Space Jacobian

- **Auclert, Bardóczy, Rognlie and Straub (2021)**. Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models.
- Instead of solving for the full transition, they show that the Jacobian (the derivatives of perfect-foresight) of the equilibrium is enough.
- They provide a very efficient method to compute these Jacobians, and show that by composing and inverting the Jacobians we can solve for the GE of the model very fast.
  - Check their NBER lecture notes at: here.
  - Also the notes of Jeppe Druedahl: here.
  - Python notebooks with plenty of examples are available: here.

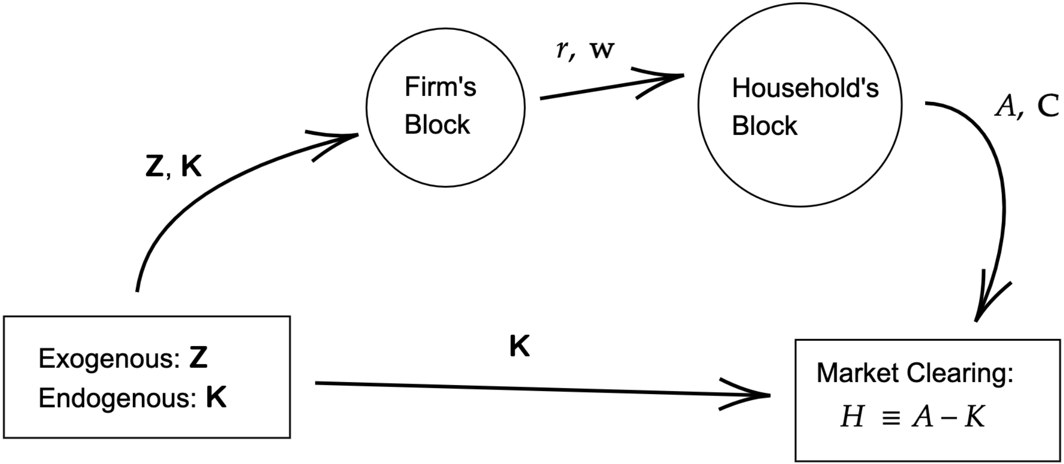
Sequence-Space Jacobian

- Their idea is that we can write the model in **blocks** and draw it as directed acyclic graphs (DAGs).
- A block is a part of the model that can be solved independently of the other parts.
  Example:
  - **Household Block** $\Rightarrow$ takes as given sequences of prices/policies (interest rates, wages, tax policies) and output sequences of aggregate consumption, savings, etc.
- Every block takes a sequence of inputs and outputs.
- The model is a combination of household block, firm block, government block, equilibrium block, etc.

Sequence-Space Jacobian

- Denote sequences of variables, e.g. Z_t, as vectors $\mathbf{Z} = (Z_0, Z_1, ...)$.
- **Example:** Krusell-Smith Model $\rightarrow$ Exogenous: $\mathbf{Z}$, Endogenous: $\mathbf{K}$.
  - **Firm's Problem**: $\mathbf{Z}, \mathbf{K} \longrightarrow \mathbf{r}, \mathbf{w}$.
  - **Household's Problem**: $\mathbf{r}, \mathbf{w} \longrightarrow \mathbf{C}, \mathbf{A}$ (where $\mathbf{C}$ and $\mathbf{A}$ are vectors of aggregate consumption and savings, e.g., $C_t = \int g_{c,t}(a,s) d\Phi_t$).
  - **Market Clearing**: $\mathbf{A}, \mathbf{K} \longrightarrow \mathbf{H} \equiv \mathbf{A} - \mathbf{K}$ (assets mkt clearing, alternatively we could have used the goods mkt).
- **Equilibrium:** There is a sequence $\mathbf{K}$, that clears the market, $\mathbf{H} = 0$, in all periods t given the sequence of exogenous variable $\mathbf{Z}$.

Block Representation of Krusell-Smith Model



Capital Response to Shocks

- Goal is to solve for market equilibrium given a sequence of exogenous shocks. In our example: $\mathbf{H} \equiv \mathbf{A} - \mathbf{K}$.
  - The sequence of aggregate savings, $\mathbf{A} = (A_0, A_1, ...)$, is a function of the entire sequences of interest rate, $\mathbf{r}$, and, $\mathbf{w}$. Further, $\mathbf{r}$, and wage, $\mathbf{w}$ are functions of the sequences of shock, $\mathbf{Z}$, and capital, $\mathbf{K}$.
  - Also, for every t, aggregate savings is a function of the **entire sequences** $\mathbf{Z}$ and $\mathbf{K}$. Then:

$$A_t(\mathbf{r}, \mathbf{w}) = A_t(\mathbf{Z}, \mathbf{K}) \tag{3}$$

- The equilibrium condition in period t is:

$$H_t(\mathbf{Z}, \mathbf{K}) = A_t(\mathbf{Z}, \mathbf{K}) - K_t$$

- The sequences of equilibrium conditions are: $\mathbf{H}(\mathbf{Z}, \mathbf{K}) = \mathbf{A}(\mathbf{Z}, \mathbf{K}) - \mathbf{K}$.

Capital Response to Shocks

- Auclert et al (2021) $\Rightarrow$ we don't need to solve for the entire equilibrium sequence to recover the response of $\mathbf{K}$ to $\mathbf{Z}$. Just need to look **Jacobians**.

- From the implicit function theorem, the linear impulse response of $\mathbf{K}$ to a transitory technology shock $d\mathbf{Z} = (dZ_0, dZ_1, ...)'$ is:

$$d\mathbf{K} = \mathbf{H}_{\mathbf{K}}^{-1}\mathbf{H}_{\mathbf{Z}} d\mathbf{Z}$$

  where $\mathbf{H}_{\mathbf{K}}$ and $\mathbf{H}_{\mathbf{Z}}$ are the Jacobians of $\mathbf{H}$ with respect to $\mathbf{K}$ and $\mathbf{Z}$, evaluated at the steady state.

- Once we have $d\mathbf{K}$, we can easily compute the response of other variables.

The Jacobians

- To compute $\mathbf{H_K}$ and $\mathbf{H_Z}$, we may have to use the chain-rule.

- For example, the eq. response to $\mathbf{Z}$ is the response of $\mathbf{A}$ to changes in $\mathbf{r}$ and, $\mathbf{w}$, which further respond to $\mathbf{Z}$. We can write as a composite of Jacobians:

$$\mathbf{H_Z} = \mathbf{J}^{A,r} \cdot \mathbf{J}^{r,Z} + \mathbf{J}^{A,w} \cdot \mathbf{J}^{w,Z}$$

  where $\mathbf{J}^{A,r}$ is the Jacobian of $\mathbf{A}$ to $\mathbf{r}$, and so on.

- The Jacobians of $\mathbf{H}$ are just the chain-rule of each **model blocks' Jacobians** ($\mathbf{J}$).

The Jacobians

- What the Jacobians look like? Depends how complicated are model blocks.
- Some are very simple, some are complicated. The "Representative firm block" is simple.
- **Example:** $\mathbf{w}$ only depends on the contemporaneous $\mathbf{Z}$.
  - $w_t = (1-\alpha)Z_t\left(\frac{K_t}{N_t}\right)^{\alpha}$. Then, the Jacobian is:

$$\mathbf{J}^{w,Z} = \begin{bmatrix} \frac{\partial w_0}{\partial Z_0} & \frac{\partial w_0}{\partial Z_1} & \cdots & \frac{\partial w_0}{\partial Z_T} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial w_T}{\partial Z_0} & \frac{\partial w_T}{\partial Z_1} & \cdots & \frac{\partial w_T}{\partial Z_T} \end{bmatrix} = \begin{bmatrix} (1-\alpha)\left(\frac{K_0}{N_0}\right)^{\alpha} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & (1-\alpha)\left(\frac{K_T}{N_T}\right)^{\alpha} \end{bmatrix}$$

  - Note that we can exploit the sparsity of the matrix.

Simple Jacobian

- $\mathbf{J}^{Y,Z} = K_t^{\alpha} N_t^{1-\alpha}$ in the diagonal.

```
J_{Y,Z}:
 [[2.28364649 0.         0.         0.         0.        ]
 [0.         2.28364649 0.         0.         0.        ]
 [0.         0.         2.28364649 0.         0.        ]
 [0.         0.         0.         2.28364649 0.        ]
 [0.         0.         0.         0.         2.28364649]]
```

The Jacobians

- The household Jacobian is complicated. Since the EE is forward looking, future shocks are anticipated by the household..
- **Example:** $\mathbf{A}$ depends on the entire path of $\mathbf{w}$.
  - Household changes its behavior in time t, once she understands her earnings change in time $t+s$.
  - Since A_t is aggregate savings, we just need that *some* households change their behavior to change A_t.

$$\mathbf{J}^{A,w} = \begin{bmatrix} \frac{\partial A_0}{\partial w_0} & \frac{\partial A_0}{\partial w_1} & \cdots & \frac{\partial A_0}{\partial w_T} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial A_T}{\partial w_0} & \frac{\partial A_T}{\partial w_1} & \cdots & \frac{\partial A_T}{\partial w_T} \end{bmatrix}$$

  - Matrix is not sparse anymore.

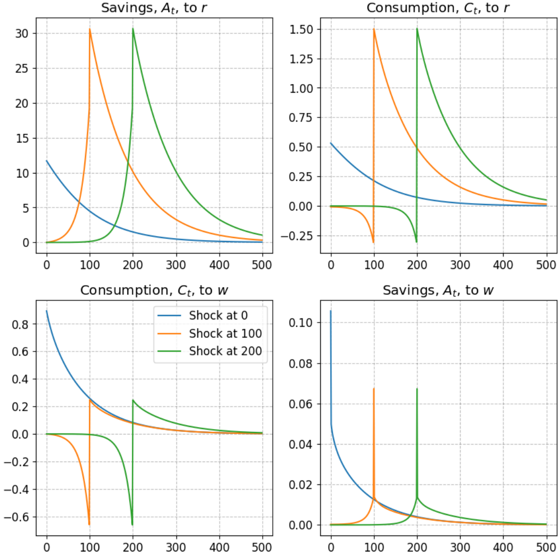
HA Jacobian

- $\mathbf{J}^{C,r}$ has intertemporal effects.

```
J_{C,r}:
 [[ 0.53172749 -1.22045882 -1.14979648 -1.08450169 -1.0237652 ]
 [ 0.52752393  0.600944   -1.15779184 -1.0919005  -1.03060995]
 [ 0.52334341  0.59396873  0.66245358 -1.10087911 -1.03888918]
 [ 0.51918469  0.58763628  0.65396671  0.71839458 -1.04871115]
 [ 0.51504555  0.5815717   0.64610874  0.70874635  0.76968641]]
```

- Each consumption response ($\frac{\partial C_i}{\partial r_j}$) is an element of the matrix:
  - If the increase of r happened in the past ($j \leq i$): consumption increases $\Rightarrow$ wealth effect changes the distribution.
  - If the increase of r will happen in the future ($j > i$): consumption decreases (savings increase) $\Rightarrow$ substitution effect.

HA Jacobians



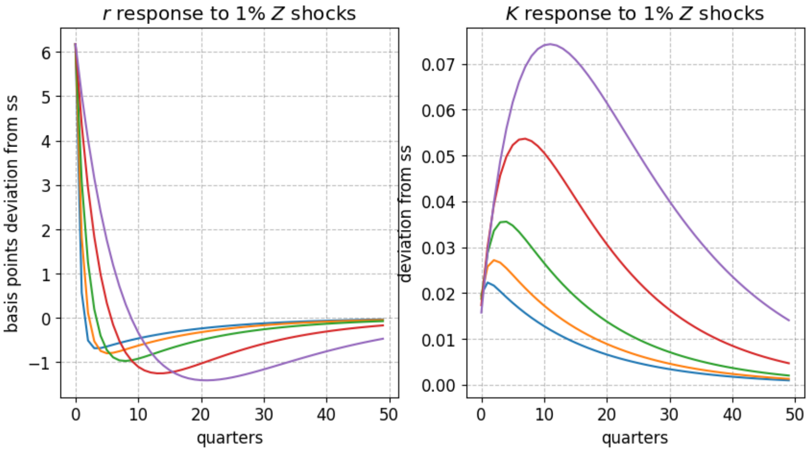
Fake News Algorithm

- **Problem:** Computing the Jacobians can be very costly $\Rightarrow$ It requires backward (policy function) and forward (distribution) for every $\mathbf{J}$.

- Auclert et al (2021) develops an algorithm based on "news shocks" (i.e., learning today that future income increases) $\Rightarrow$ Fake News Algorithm.

- **Intuition:**
  - Only the difference between two periods matter (not the actual t) for policy functions $\Rightarrow$ a single backward iteration is sufficient.
  - For the effect through the distribution, they use a "Fake News" shock: a shock in period s announced in $t = 0$ but retracted at $t = 1$.
  - Using tedious algebra and the chain-rule they can construct all the Jacobians fast.

Impulse Response Function



Non-linear Solutions

- The Jacobians give a linearized IRF. They can be imprecise for large shocks or in models with aggregate non-linearities.

- The package also give an algorithm to compute the **nonlinear perfect foresight dynamics** (i.e., the MIT shock).

- The idea is to use the fact that an equilibrium must solve: $H(K, Z) = 0$, iterate in a sequence of K^j, where j is the guess of K, and update using:

$$K^{j+1} = K^j - H_K(K_{ss}, Z_{ss})^{-1} H(K^j, Z)$$

- Note this is very similar to a Newton Algorithm, which in practice has very fast convergence.

Sequence-Space Jacobian

- Once we have the Jacobians of each model block, we can compute the response to any type of shocks, IRF, or transition dynamics for a new SS.
- The key is to compute the Jacobians efficiently.
- The algorithm allows us to solve even very complex HANK models.
- It can also be applied to more general models (entry-exit, discrete choices, etc), but some details must be taken care of.
- **Limitations**:
  - $\Rightarrow$ models where the Bellman equation depends directly on the distribution (e.g., wage posting search models).
  - $\Rightarrow$ solving the stationary equilibrium can be costly in some models, must apply some tricks to speed up this step.

Application: Krueger, Mitman and Perri

- **Question:** How important is household heterogeneity for the amplification and propagation of macroeconomic shocks?
- Focus on the US Great Recession of 2007–09.
- Heterogeneity: earnings, wealth, and household preferences.
- Consequences for cross-sectional inequality in disposable income and consumption expenditures.

Krueger, Mitman and Perri (2016, Handbook of Macro)

Method:

- Summarize empirical facts about the joint distribution of income, wealth, and consumption before and during the great recession.
- Compute various versions of the HA model with aggregate uncertainty and study its cross-sectional and dynamic properties.
  - Simple version of the model "replicates" the results of representative agent model.
  - Model extension with life-cycle, unemployment insurance and social security does a much better job.
- Study the impact of social insurance policies.

Empirical Evidence: Levels

- **Data**: PSID (2004, 2006, 2008, and 2010). New version covers income, wealth, and consumption.

Table 2 PSID Households across the net worth distribution: 2006

	% Share of:			% Expend. Rate		Head's	
NW Q	**Earn.**	**Disp. *Y***	**Expend.**	**Earn.**	**Disp. *Y***	**Age**	**Edu. (yrs)**
Q1	9.8	8.7	11.3	95.1	90.0	39.2	12
Q2	12.9	11.2	12.4	79.3	76.4	40.3	12
Q3	18.0	16.7	16.8	77.5	69.8	42.3	12.4
Q4	22.3	22.1	22.4	82.3	69.6	46.2	12.7
Q5	37.0	41.2	37.2	83.0	62.5	48.8	13.9
	Correlation with net worth						
	0.26	0.42	0.20				

Empirical Evidence: Changes

Table 3 Annualized changes in selected variables across PSID net worth

	Net worth[a]				Disp. *Y* (%)		Cons. Exp.(%)		Exp. Rate (pp)	
	(1) 04–06		(2) 06–10		(3) 04–06	(4) 06–10	(5) 04–06	(6) 06–10	(7) 04–06	(8) 06–10
All	**15.7**	**44.6**	**−3.0**	**−10**	**4.1**	**1.2**	**5.6**	**−1.3**	**0.9**	**−1.6**
NW Q										
Q1	NA	12.9	NA	6.6	7.4	6.7	7.1	0.6	−0.2	−4.2
Q2	121.9	19.5	24.4	3.7	6.7	4.1	7.2	2	0.3	−1.3
Q3	32.9	23.6	4.3	3.3	5.1	1.8	9	0	2.3	−1.1
Q4	17.0	34.7	1.7	3.8	5.0	1.7	5.9	−1.5	0.5	−2
Q5	11.6	132.2	−4.9	−68.4	1.8	−1.2	2.7	−3.5	0.5	−1.4

[a]The first figure is the percentage change (growth rate), the second is the change in 000's of dollars.

- **Last column**: saving rates increase relatively more for wealth-poor households during the recession.

A Business Cycle Model with HH Heterogeneity

Ingredients:

- Idiosyncratic individual shocks + incomplete markets a la Aiyagari-Hugget.
- Aggregate Shocks in the spirit of the Real Business Cycle literature.
- 2 stages life cycle: young (workers) and old (retiree).
- Ex-ante heterogeneity in β.
- Government policy: unemployment insurance and social security.

Production Technology

- Aggregate production function is Cobb-Douglas over capital and labor:

$$Y = Z^* K^\alpha N^{1-\alpha} \tag{4}$$

- $Z^* \equiv Z C^\omega$, where $\omega \geq 0$.

- **Aggregate shock**: Z follows a 2-state Markov with transition matrix $\pi(Z'|Z)$:
  - $Z \in \{Z_l, Z_h\}$. Z_l: recession, Z_h : normal times.

- **Demand externality**: C^ω
  - If $\omega = 0$, standard neoclassical production function.
  - If $\omega > 0$, production is partially determined by demand.

Households

- Standard utility over consumption $u(c)$, they cannot borrow, $a' \geq 0$.
- Ex-ante and fixed heterogeneous discount factor $\beta \in B$.
- Two idiosyncratic states:
  - Employment: $s \in \{e, u\}$. Transition matrix depends on aggregate state: $\pi(s'|s, Z', Z)$.
  - Income: γ. Transition matrix is independent of the aggregate state: $\pi(\gamma'|\gamma)$.
- Stochastic life cycle:
  - households are born as workers and with probability $1 - \theta$ they retire;
  - after retiring, they receive pensions and with probability $1 - \nu$ they die.

Government Policy

- Pensions and unemployment benefits are financed using proportional labor taxes.
- Government runs a balance budget system every period.
- **Unemployment insurance**:
  - Pays a fraction $\rho \in [0, 1)$ of the household potential income: $b = \rho w \gamma$.
  - Financed with tax rate, $\tau(Z)$. The tax adjusts to maintain the budget balanced. In recessions, the tax rate increases.
- **Pension benefits**:
  - Financed with a fixed social security contribution τ_{ss}.
  - Pension benefits, b_{ss}, adjust to maintain the budget balanced. In recessions, the pension decreases.

Household Bellman Equation: Retiree

$$V_R(a,\beta;Z,\Phi) = \max_{c,\ a'\geq 0}\left\{u(c) + \nu\beta\sum_{Z'\in Z}\pi(Z'|Z)V_R(a',\beta;Z',\Phi')\right\}$$

$$\begin{aligned} \text{s.t} \qquad & c + a' = b_{ss}(Z,\Phi) + (1 + r(Z,\Phi) - \delta)a/\nu \\ & \Phi' = H(Z,\Phi,Z') \end{aligned}$$

where Φ is the distribution of agents in the economy.

- Individual state: (a,β), aggregate state: (Z,Φ).
- $H()$: Rational expectations function. The agents correctly forecast the next period distribution, given the state of the economy.

Household Bellman Equation: Worker

$$V_w(s,\gamma,a,\beta;Z,\Phi) = \max_{c,\ a'\geq 0}\Bigg\{\{u(c)+\beta\sum_{(Z',s',\gamma')\in(Z,s,\gamma)}\pi(Z'|Z)\pi(s'|s,Z',Z)\pi(\gamma'|\gamma)$$

$$\times\left[\theta V_W(s',\gamma',a',\beta;Z',\Phi')+(1-\theta)V_R(s',\gamma',a',\beta;Z',\Phi')\right]\Bigg\}$$

$$\text{s.t} \qquad c+a' = (1-\tau(Z)-\tau_{ss})\gamma w(Z,\Phi)[1-(1-\rho)\mathbf{1}_{s=u}]+(1+r(Z,\Phi)-\delta)a$$

$$\Phi' = H(Z,\Phi,Z')$$

- Individual state: (a,s,γ,β), aggregate state: (Z,Φ).
- Employed earnings: γw; unemployed earnings $\rho\gamma w$.

Equilibrium

- Prices are given by FOCs of firm's problem:

$$w(Z,\Phi) = Z\alpha \left(\frac{K(Z,\Phi)}{N(Z)}\right)^{1-\alpha} \quad \text{and} \quad r(Z,\Phi) = Z(1-\alpha)\left(\frac{N(Z)}{K(Z,\Phi)}\right)^{\alpha}$$

  where aggregate employment $N(Z)$ is given by the distributions of the Markov process (which depends on Z).

- Asset market clears:

$$K(Z,\Phi) = \int a d\Phi$$

- The distribution evolves according to the function: $\Phi' = H(Z,\Phi,Z')$. In equilibrium, this function is *consistent* with the individual decisions.

Krueger, Mitman and Perri: Models

Table 5 Taxonomy of different versions of the model used in the chapter

Name	Discounting	Techn.	Soc. Ins.
KS	$\beta = \overline{\beta}$	$\omega = 0$	$\rho = 1\%$
Het. β	$\beta \in [\overline{\beta} - \epsilon \, \overline{\beta} + \epsilon]$	$\omega = 0$	$\rho = 50\%$
Het. β	$\beta \in [\overline{\beta} - \epsilon \, \overline{\beta} + \epsilon]$	$\omega = 0$	$\rho = 10\%$
Dem. Ext.	$\beta \in [\overline{\beta} - \epsilon \, \overline{\beta} + \epsilon]$	$\omega > 0$	$\rho = 50\%$

- **KS**: Basic model, very similar to Krusell-Smith (1998);
- **Benchmark model**: second row, calibrated to match the US economy.

Calibration

- Model calibrated to quarterly data. $\alpha = 0.36$, $\delta = 0.025$, $u(c) = \log(c)$.
- Z duration and GDP drop of a "severe recession".
- s separation and job-finding rates, γ comes from a persistent-transitory process estimated from PSID.
- θ, ν: a working period of 40 years and a retirement period of 15 years.
- β from uniform distribution: match Gini for the wealth distribution.
- Policy: unemployment benefits of 50%, $\rho = 0.5$. Pension of 40% of avg. wage.

Evaluating the Model: Wealth Distribution

Table 6 Net worth distributions: Data vs models

	Data		Models	
% Share held by:	**PSID, 06**	**SCF, 07**	**Bench**	**KS**
Q1	−0.9	−0.2	0.3	6.9
Q2	0.8	1.2	1.2	11.7
Q3	4.4	4.6	4.7	16.0
Q4	13.0	11.9	16.0	22.3
Q5	82.7	82.5	77.8	43.0
90–95	13.7	11.1	17.9	10.5
95–99	22.8	25.3	26.0	11.8
*T*1%	30.9	33.5	14.2	5.0
Gini	0.77	0.78	0.77	0.35

- Benchmark matches the wealth distribution; but fails in the top 1%. KS fails.

Evaluating the Model: Joint Distribution

Table 8 Selected variables by net worth: Data vs models

	% Share of:						% Expend. rate			
	Earnings		Disp. Y		Expend.		Earnings		Disp. Y	
NW Q	Data	Mod	Data	Mod	Data	Mod	Data	Mod	Data	Mod
Q1	9.8	6.5	8.7	6.0	11.3	6.6	95.1	96.5	90.0	90.4
Q2	12.9	11.8	11.2	10.5	12.4	11.3	79.3	90.3	76.4	86.9
Q3	18.0	18.2	16.7	16.6	16.8	16.6	77.5	86.0	69.8	81.1
Q4	22.3	25.5	22.1	24.3	22.4	23.6	82.3	87.3	69.6	78.5
Q5	37.0	38.0	41.2	42.7	37.2	42.0	83.0	104.5	62.5	79.6
	Correlation with net worth									
	0.26	0.46	0.42	0.67	0.20	0.76				

- Qualitatively close to the data, but quantitative a bit far; wealth poor consuming too little, and the wealth rich consuming too much.

Evaluating the Model: Dynamics in Normal Times

Table 9 Annualized changes in selected variables by net worth in normal times (2004-06): Data vs model

	Net worth (%)		Disp. Y (%)		Expend (%)		Exp. Rate (pp)	
NW Q	**Data**	**Model**	**Data**	**Model**	**Data**	**Model**	**Data**	**Model**
Q1	NaN	44	7.4	7.2	7.1	6.7	−0.2	−0.4
Q2	122	33	6.7	3.1	7.2	3.6	0.3	0.5
Q3	33	20	5.1	1.6	9	2.5	2.3	0.8
Q4	17	9	5	0.5	5.9	1.7	0.5	1.2
Q5	12	3	1.8	−1.0	2.7	0.5	0.5	1.4
All	16	5	4.1	0.7	5.6	1.8	0.9	0.7

- Slightly too much downward and upward mobility on income, but in general good job.

Evaluating the Model: Dynamics in Recession

Table 10 Annualized changes in selected variables by net worth in a severe recession: Data vs model

	Net worth (%)		Disp. Y (%)		Expend. (%)		Exp. rate (pp)	
NW Q	Data	Model	Data	Model	Data	Model	Data	Model
Q1	NaN	24	6.7	4.9	0.6	4.5	−4.2	−0.4
Q2	24	15	4.1	0.3	2.0	1.2	−1.3	0.8
Q3	4	8	1.8	−2.4	0.8	0.0	−1.1	2.2
Q4	2	4	1.7	−4.0	−1.7	−1.5	−2.0	3.2
Q5	−5	−1	−1.2	−6.4	−3.7	−3.5	−1.4	4.6
All	−3	1	1.2	−3.7	−1.3	−0.8	−1.6	2.0

- Consumption-savings in the recession: $\downarrow$ savings because of consumption smoothing; $\uparrow$ savings because of precautionary savings.
  - In the model: the first is stronger for the richer, but the latter is stronger for the poorer.

Aggregate Shock: Krusell-Smith vs Representative Agent

- The Krusell-Smith economy is remarkably similar to the representative agent in the aggregate.

- **Intuition**: without too many constrained agents, the HA economy behaves as a RA.

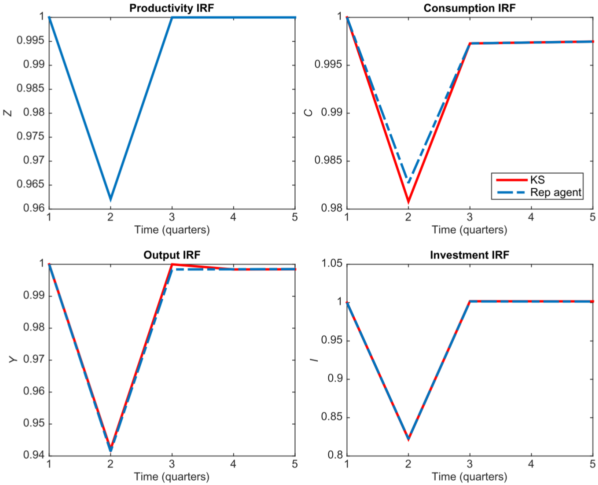
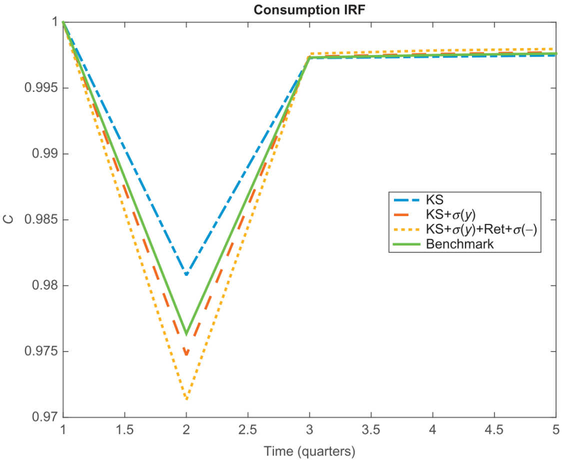


Fig. 3 Impulse response functions (IRF) to aggregate technology shock in KS and RA economies.

Aggregate Shock: All models

- Benchmark generates a larger drop in consumption than KS economy.
- Largely accounted by income risk on top of employment risk.
- Recall that benchmark economy has high unemployment benefits.



Differences between KS and Benchmark Economy

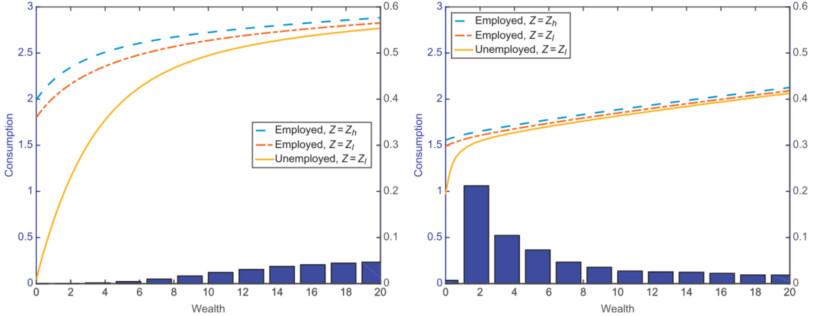


Fig. 5 Consumption function and wealth distribution: Krusell–Smith (left panel) and benchmark (right panel).

- Benchmark generates a larger drop in consumption because it has a larger share of low wealth households.
- The low wealth consumes more in the benchmark because of unemployment benefits.

The Role of Unemployment Insurance

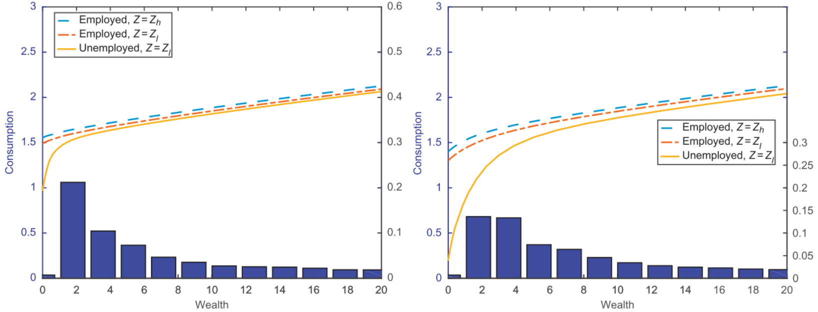


Fig. 10 Consumption function and wealth distribution: Benchmark (left panel) and low UI (right panel).

- Unemployment benefits help the low wealth poor to consume in bad times.
- Aggregate consumption falls much more in recessions without UI.

Demand Externality

- Keynesian flavor increases the size of the recession.

- Lots of wealth poor ⇒ large drop in consumption ⇒ demand externality ⇒ further drops output.

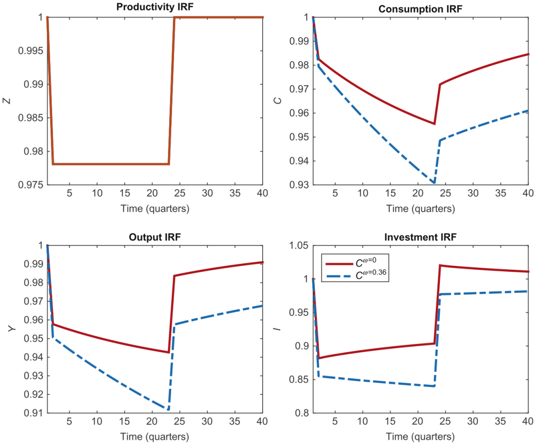


Fig. 14 Impulse response to identical aggregate technology shock: Comparison between economies with and without demand externality.

Conclusion: Krueger, Mitman and Perri

- Simple heterogeneity a la Krusell-Smith/Aiyagari is not enough to generate differences from the representative agent model.
- Other ingredients should be added to get a meaningful wealth distribution.
- Low wealth agents are key to getting the larger fall in consumption.
- Unemployment insurance attenuates the fall of aggregate consumption.
- Demand externality further increases the recession: motivation to include a proper microfoundation of the demand effect.

Where to go now?

- **Income Risk and Business Cycles**: Bayer et al (2019, ECTA), McKay (2017, JME).
- **Search Frictions and Unemployment**: Ravn and Sterk (2017, JME), Nakajima (2012, JME; 2012, IER).
- **Precautionary Savings over the Cycles**: Challe and Ragot (2016, EJ), Heathcote and Perri (2017, ReStud).
- **Credit Crunch and Housing**: Guerrieri and Lorenzoni (2017, QJE), Kaplan, Mitman and Violante (2020, JPE).
- **Automatic Stabilizers**: McKay and Reis (2017, ECTA; 2021, ReStud).
- **Trends in Inequality**: Heathcote et al (2010, JPE), Heathcote, Perri and Violante (2020, RED).
- Transition dynamics of all questions we saw before.